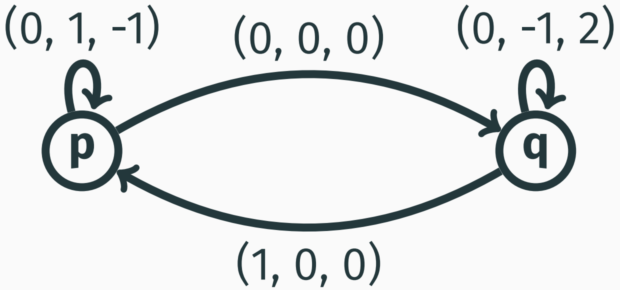
Affine Extensions of Integer Vector Addition Systems with States

Michael Blondin Christoph Haase Filip Mazowiecki

UNIVERSITÉ DE SHERBROOKE
UNIVERSITY OF OXFORD
université de BORDEAUX
Technical University of Munich

Vector addition systems with states (VASS)



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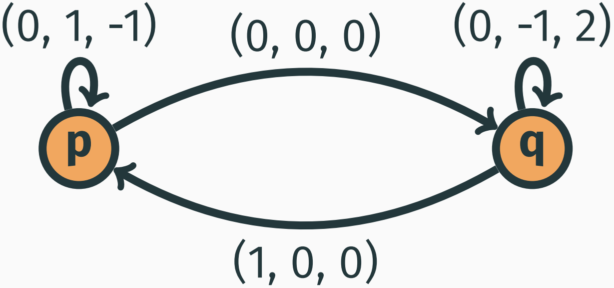
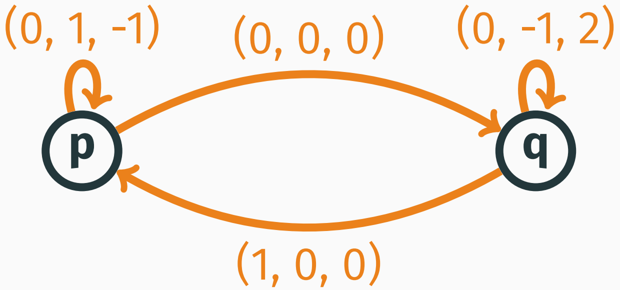


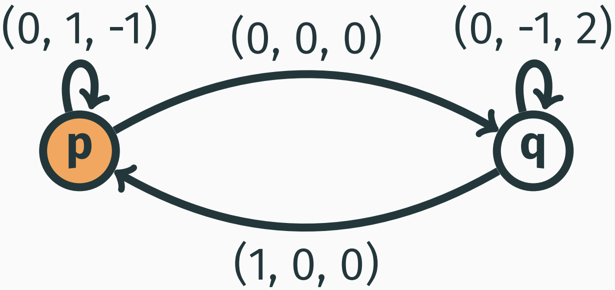
Control-states

Vector addition systems with states (VASS)



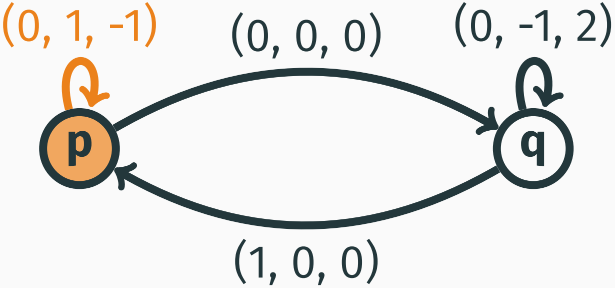
Transitions

Vector addition systems with states (VASS)



p(0, 0, 1)

Vector addition systems with states (VASS)



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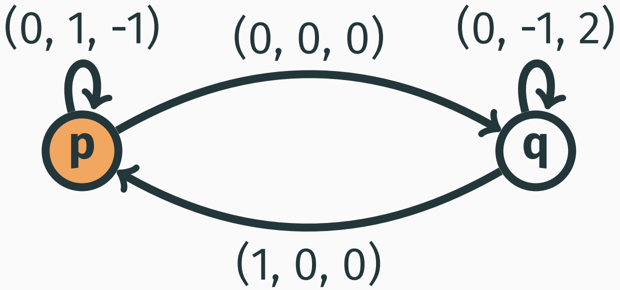

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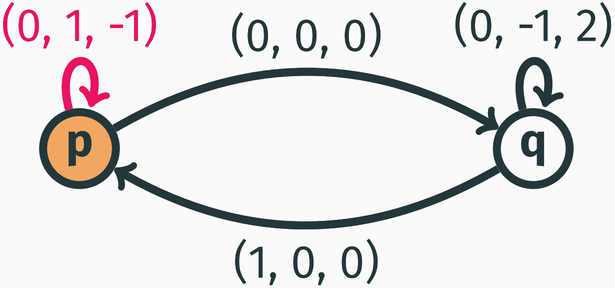

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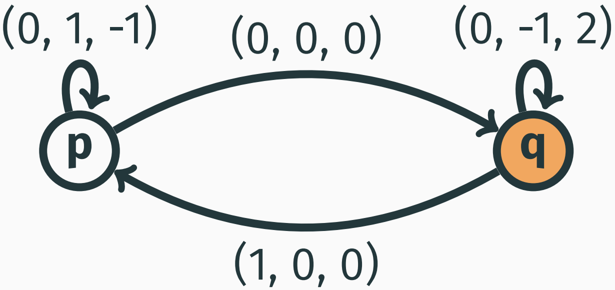
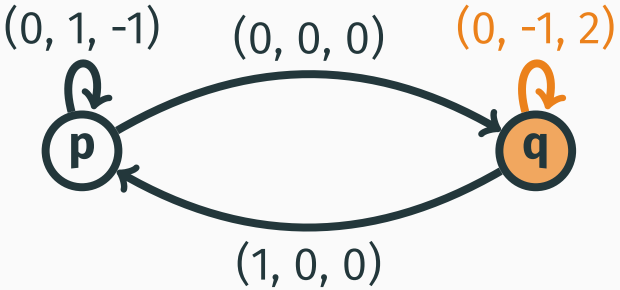


q(0, 1, 0)

Vector addition systems with states (VASS)



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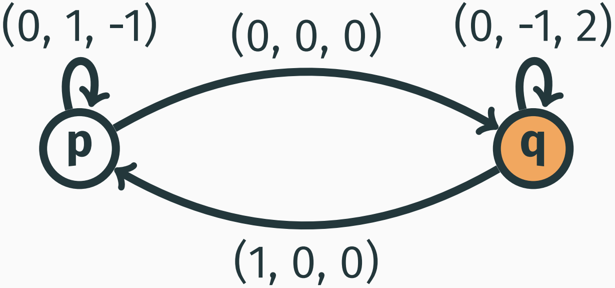

q(0, 0, 2)

Vector addition systems with states (VASS)

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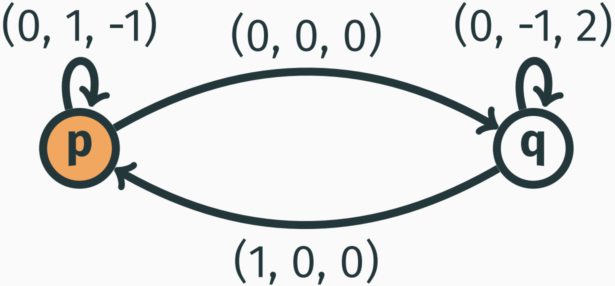
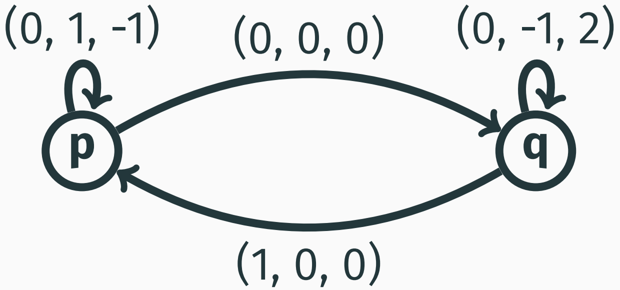
Vector addition systems with states (VASS)



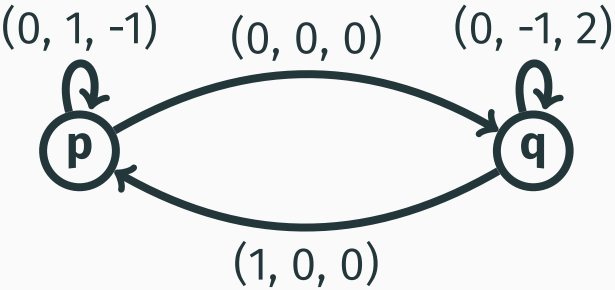
p(1, 0, 2)

Vector addition systems with states (VASS)



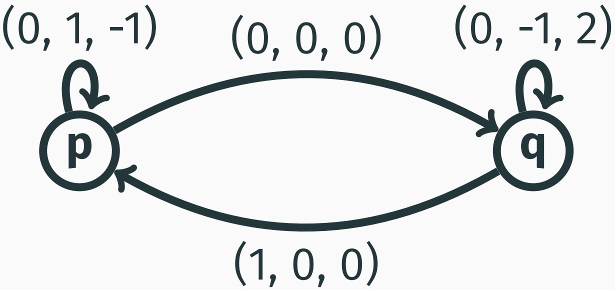
$$p(0, 0, 1) \xrightarrow{*}_{\mathbb{N}} p(1, 0, 2)$$

Vector addition systems with states (VASS)



$$p(0, 0, 1) \xrightarrow{*}_{\mathbb{N}} p(x, y, z) \iff 0 < y + z \leq 2^x$$

Vector addition systems with states (VASS)



Reachability: $p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{N}} q(\boldsymbol{v})$?

Coverability: $p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{N}} q(\geq \boldsymbol{v})$?

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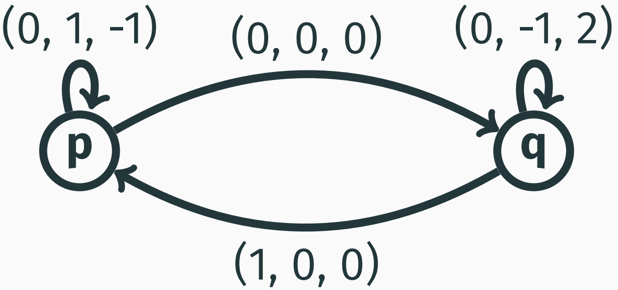

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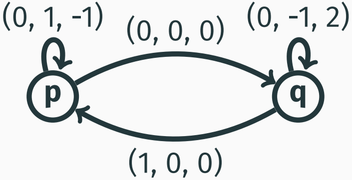
Vector addition systems with states (VASS)

- Concurrent programs
- Protocols
- Business processes
- Biological processes
- ⋮

correct? →



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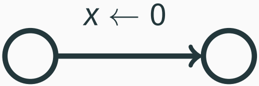
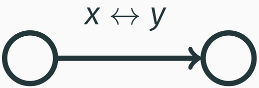
Common operations used for modeling:

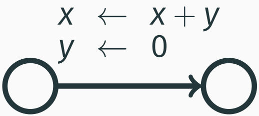
Reset



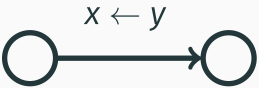
Swap



Transfer



Copy



Common operations used for modeling:

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$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

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Copy

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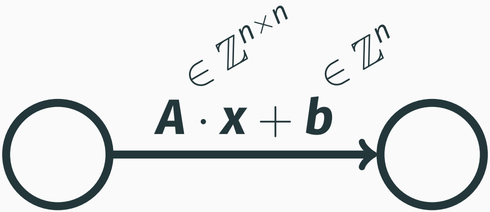
All affine transformations!

Affine VASS:



$\in \mathbb{Z}^{n\times n}$
$\in \mathbb{Z}^{n}$
$\boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{b}$

Complexity of reachability and coverability

	No extensions	+ Resets	+ Transfers
$\xrightarrow{*}_{\mathbb{N}}$	EXPSPACE-hard (Lipton ’76) $\in$ h.-Ackermann (Leroux, Schmitz ’15)	Undecidable (Araki, Kasami ’76)	
$\xrightarrow{*}_{\mathbb{N}} \geq$	EXPSPACE-complete (Lipton ’76, Rackoff ’78)	Ackermann-complete (Schnoebelen ’02, Figueira *et al.* ’11)	

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Intractable!

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- Can be alleviated by using an over-approximation of $\xrightarrow{*}_{\mathbb{N}}$
- Successful in practice, *e.g.* Esparza *et al.* CAV'14, B. *et al.* TACAS'16, Geffroy *et al.* RP'16, Athanasiou *et al.* IJCAR'16
- We consider $\mathbb{Z}$-VASS: counters allowed to drop below 0

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$\xrightarrow{*}_{\mathbb{Z}}$ $\xrightarrow{*}_{\mathbb{Z}} \geq$	NP-complete (Haase, Halfon '14)		?

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$\xrightarrow{*}_{\mathbb{Z}}$ $\xrightarrow{*}_{\mathbb{Z}} \geq$	NP-complete (new proof)		PSPACE-complete

Our contribution

- Any affine $\mathbb{Z}$-VASS with finite matrix monoid can be translated into an equivalent $\mathbb{Z}$-VASS
- Reachability relation of such affine $\mathbb{Z}$-VASS is semilinear
- Classification of complexity w.r.t. extensions

Complexity of reachability and coverability

Related work

- Finkel and Leroux (FSTTCS'12)

  Accelerations of affine counter machines without control-states

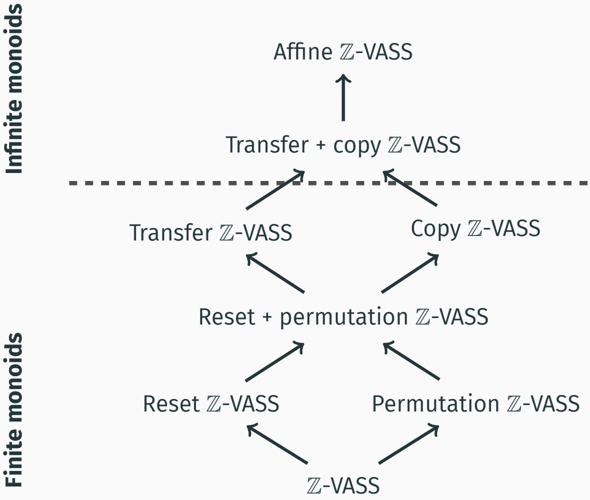
- Iosif and Sangnier (ATVA'16)

  Complexity of model checking over flat structures with guards defined by convex polyhedra

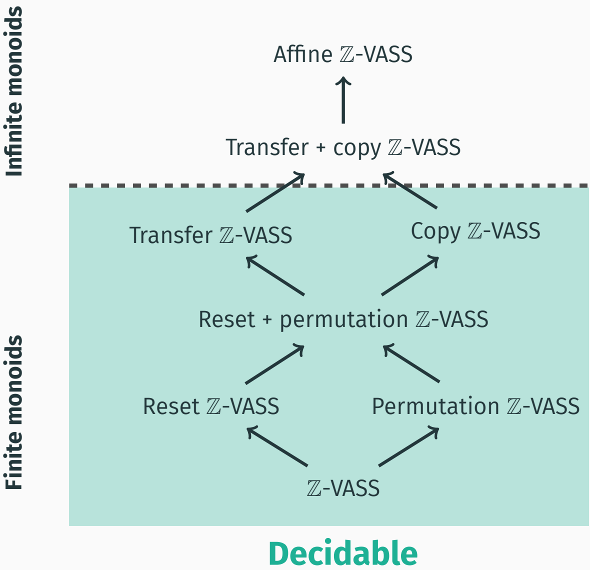
- Cadilhac, Finkel and McKenzie (IJFCS'12)

  Affine Parikh automata with finite-monoid restriction

Overview


Infinite monoids
Affine ℤ-VASS
Transfer + copy ℤ-VASS
Transfer ℤ-VASS
Copy ℤ-VASS
Reset + permutation ℤ-VASS
Reset ℤ-VASS
Permutation ℤ-VASS
ℤ-VASS
Finite monoids

Overview



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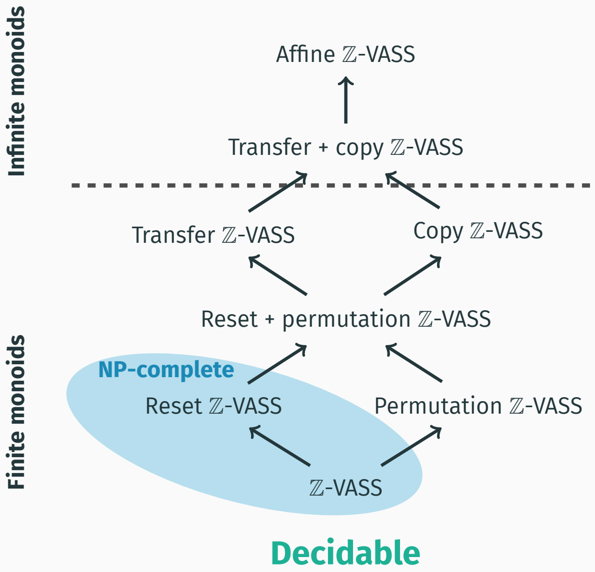
Finite monoids
Decidable

Overview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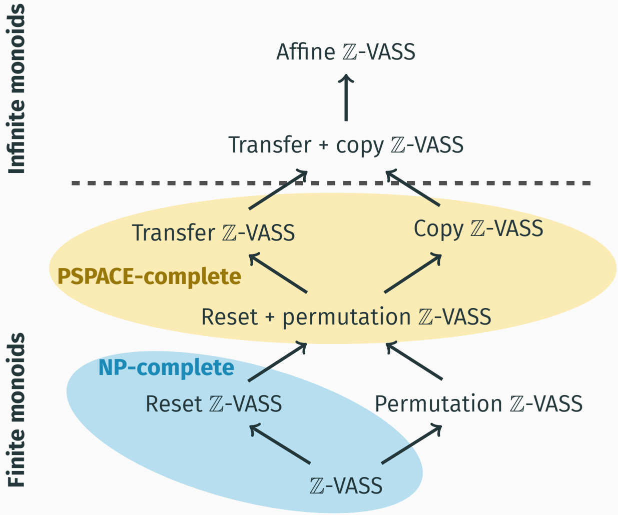
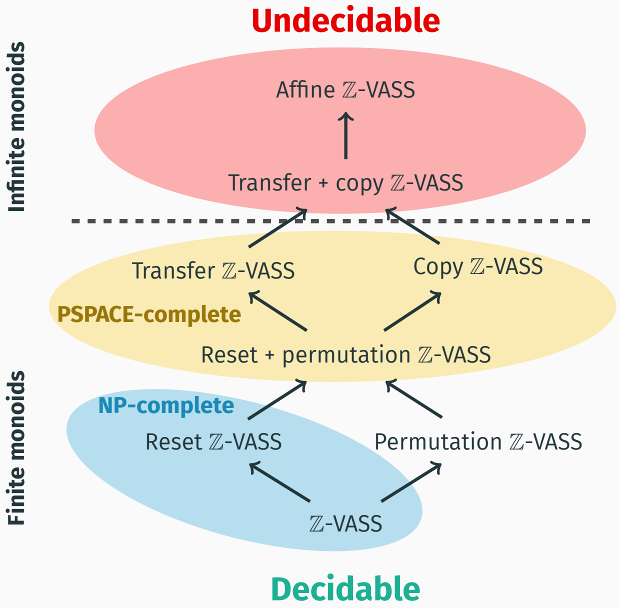


Decidable

Overview


Undecidable
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Decidable

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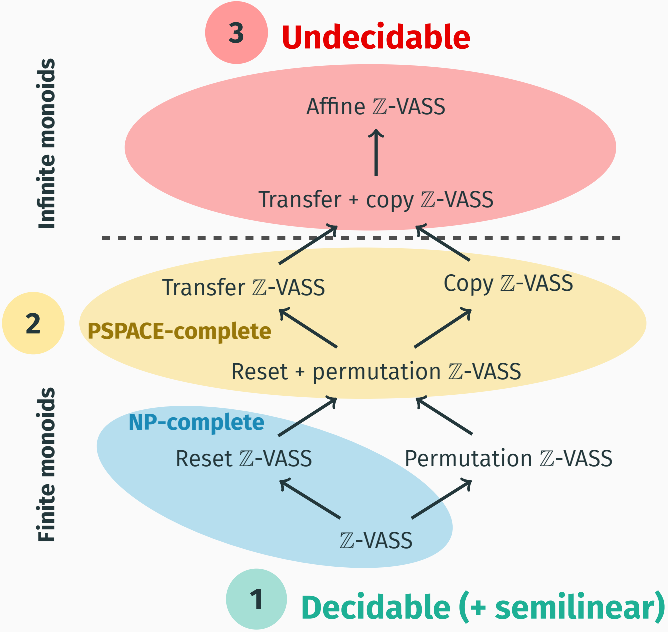
3
Undecidable
Infinite monoids
Affine ℤ-VASS
Transfer + copy ℤ-VASS
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2
PSPACE-complete
Reset + permutation ℤ-VASS
Finite monoids
NP-complete
Reset ℤ-VASS
Permutation ℤ-VASS
ℤ-VASS
1
Decidable (+ semilinear)

A few definitions

For every transition $t\colon \textcircled{p} \xrightarrow{\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{b}} \textcircled{q}$ and $\sigma \in T^*$, let

$$M_\varepsilon = \boldsymbol{I} \qquad \varepsilon(\boldsymbol{u}) = \boldsymbol{u}$$

$$M_{\sigma t} = \boldsymbol{A} \cdot M_\sigma \qquad \sigma t(\boldsymbol{u}) = \boldsymbol{A} \cdot \sigma(\boldsymbol{u}) + \boldsymbol{b}$$

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Matrix monoid

$$\mathcal{M}_\mathcal{V} = \{M_w : w \in T^*\}$$

From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

Theorem

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $\mathcal{M}_{\mathcal{V}}$ is finite, then $\exists$ $\mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{Z}} q(\boldsymbol{v})$ *in* $\mathcal{V} \iff p(\boldsymbol{u}, \mathbf{0}) \xrightarrow{*}_{\mathbb{Z}} q(\mathbf{0}, \boldsymbol{v})$ *in* $\mathcal{V}'$
- $|\mathcal{V}'| \in \mathrm{poly}(|\mathcal{V}|, |\mathcal{M}_{\mathcal{V}}|)$

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Proof sketch

$p(\boldsymbol{u}) \xrightarrow{w}_\mathbb{Z} q(\boldsymbol{v}) \iff$

- w is a path from p to q
- $\boldsymbol{v} = w(\boldsymbol{u})$

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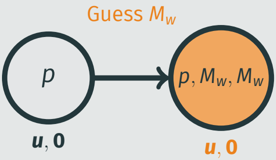

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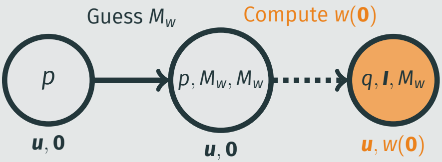

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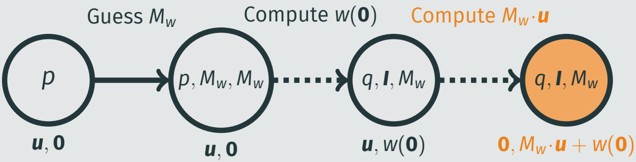

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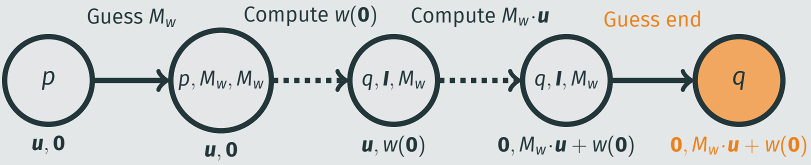
- $p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{Z}} q(\boldsymbol{v})$ *in* $\mathcal{V} \iff p(\boldsymbol{u}, \mathbf{0}) \xrightarrow{*}_{\mathbb{Z}} q(\mathbf{0}, \boldsymbol{v})$ *in* $\mathcal{V}'$
- $|\mathcal{V}'| \in \mathrm{poly}(|\mathcal{V}|, |\mathcal{M}_{\mathcal{V}}|)$

Proof sketch

$p(\boldsymbol{u}) \xrightarrow{w}_{\mathbb{Z}} q(\boldsymbol{v}) \iff$

- w is a path from p to q
- $\boldsymbol{v} = M_w \cdot \boldsymbol{u} + w(\mathbf{0})$



From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

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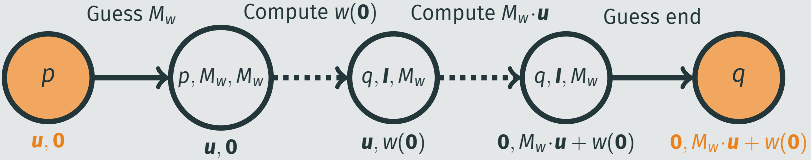

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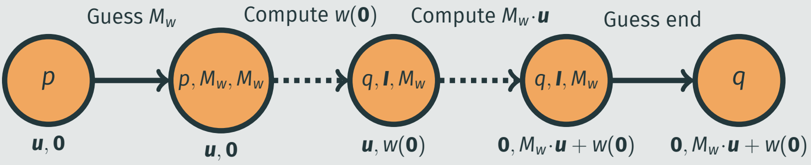

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Corollary

Reachability is decidable for affine $\mathbb{Z}$-VASS with finite matrix monoid

Semilinearity of affine $\mathbb{Z}$-VASS

Corollary

If an affine $\mathbb{Z}$-VASS has a finite monoid, then

$$\left\{(\boldsymbol{u}, \boldsymbol{v}) : p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{Z}} q(\boldsymbol{v})\right\} \text{ is semilinear}$$

Proof

Follows from our translation and

known result on $\mathbb{Z}$-VASS (Haase, Halfon RP'14)

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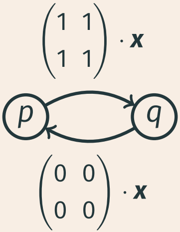
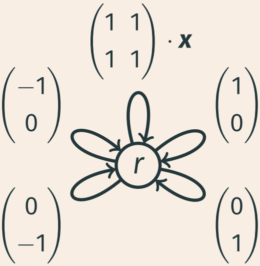
Observation

Converse is not true:





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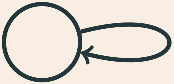
Observation — **Boigelot '98, Finkel and Leroux '02**

Converse is true for single state and single transition:



$\boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{b}$

Reachability in transfer $\mathbb{Z}$-VASS is in PSPACE

- Transfer matrix: exactly one 1 per column,
  hence $|\mathcal{M}_{\mathcal{V}}| \leq 2^n$
- Transform transfer $\mathbb{Z}$-VASS $\mathcal{V}$ into $\mathbb{Z}$-VASS $\mathcal{V}'$
  of size $\mathrm{poly}(|\mathcal{V}|, 2^n)$
- $\mathbb{Z}$-reachability has witnesses of the form $w_1^{k_1} w_2^{k_2} \cdots w_\ell^{k_\ell}$
  where $|w_1 w_2 \cdots w_\ell| \leq \mathrm{poly}(|\mathcal{V}'|)$ (B. *et al.* LICS'15)
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Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

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$$q_0\left(\overset{a}{1}\quad\overset{b}{0}\quad\overset{a}{0}\quad\overset{b}{1}\quad\cdots\quad\overset{a}{0}\quad\overset{b}{1}\right)$$

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Idea: simulate linear bounded Turing machine

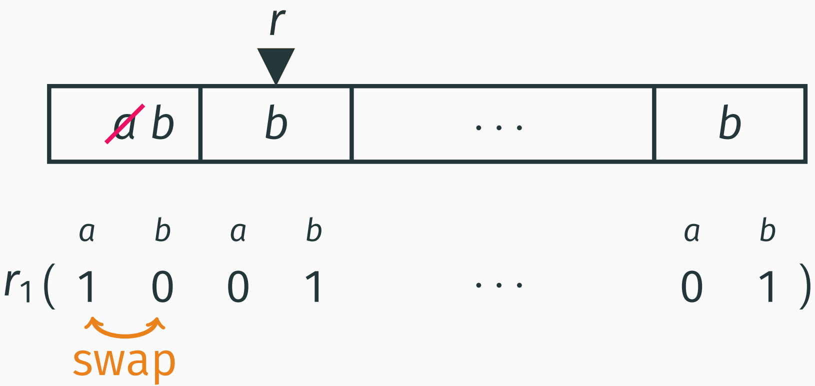
$$q_0 \begin{pmatrix} \overset{a}{1} & \overset{b}{0} & \overset{a}{0} & \overset{b}{1} & \cdots & \overset{a}{0} & \overset{b}{1} \end{pmatrix}$$

Idea: simulate linear bounded Turing machine

$$r_1 \; \begin{array}{c} \\ \left(\right. \end{array} \begin{array}{cccccccc} a & b & a & b & & & a & b \\ 1 & 0 & 0 & 1 & \cdots & & 0 & 1 \end{array} \begin{array}{c} \\ \left. \right) \end{array}$$

Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine



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Reachability in transfer ℤ-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

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reset

Reachability in transfer ℤ-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

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Simulation is faithful iff
the sum of bits is left unchanged

Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

$$\begin{matrix} & a & b & a & b & & a & b \\ r_1 (& 0 & 1 & 0 & 1 & \cdots & 0 & 1) \end{matrix}$$

Swaps and resets

can be simulated by transfers

Reachability in affine $\mathbb{Z}$-VASS is undecidable

Reduction from the Post correspondence problem

$$w_1 = \frac{10}{1} \qquad w_2 = \frac{01}{0} \qquad w_3 = \frac{1}{011}$$

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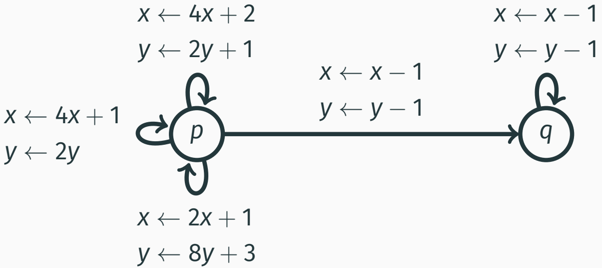

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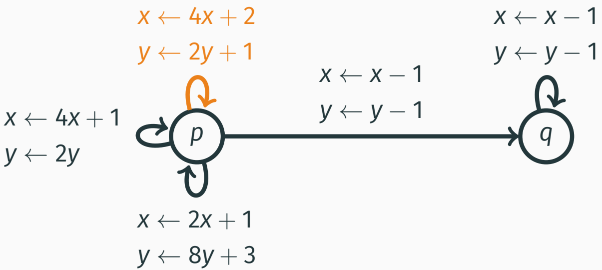

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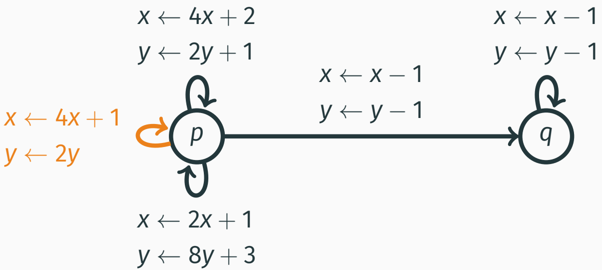

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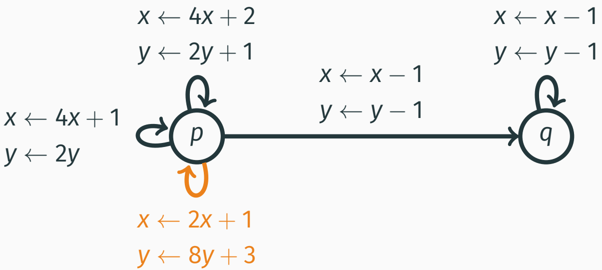

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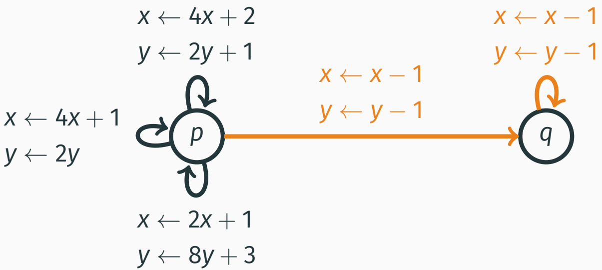

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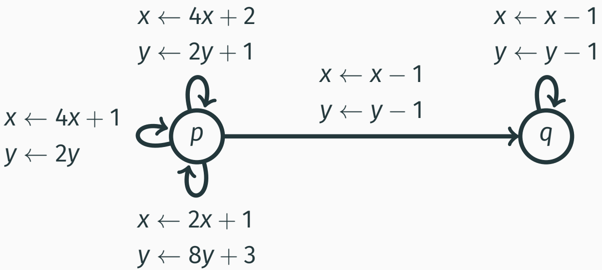

Has solution iff $p(1,1) \xrightarrow{*}_{\mathbb{Z}} q(1,1)$

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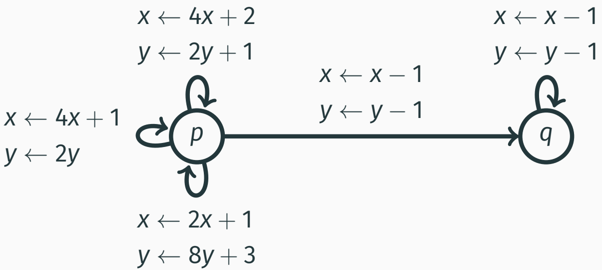

Doubling can be done with
a gadget of transfers and copies

Conclusion: summary

- Unified approach to reachability in affine $\mathbb{Z}$-VASS
- Possible to remove transformations when matrix monoid is finite
- Reachability relation of affine $\mathbb{Z}$-VASS is semilinear when monoid is finite
- Classification of complexity w.r.t. extensions

Conclusion: further work

- Complexity of reachability for permutation $\mathbb{Z}$-VASS?
- Size of matrix monoid for arbitrary affine $\mathbb{Z}$-VASS?
- Characterization of classes of infinite matrix monoids for which reachability is undecidable?

Thank you!