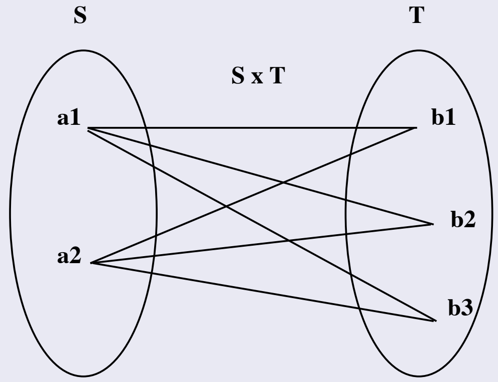
Set Theory: Basic Constructs

There are three basic constructs in set theory:

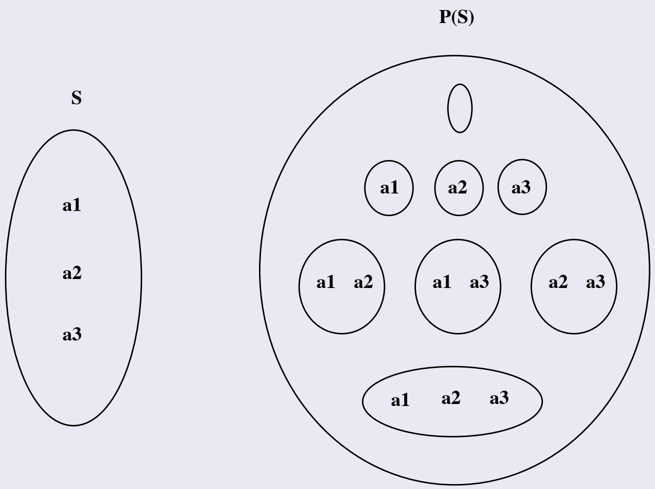
Cartesian product	$S \times T$
Power set	$\mathbb{P}(S)$
Comprehension 2	$\{\, x \mid P \,\}$

where S and T are sets, x is a variable and P is a predicate.

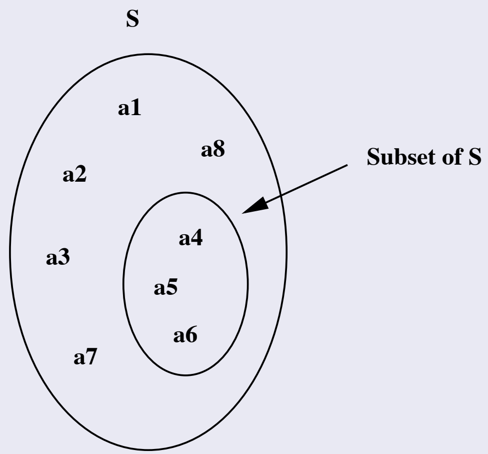
Cartesian Product



Power Set



Set Comprehension



Basic Set Operator Memberships (Axioms)

These axioms are defined by equivalences.

Left Part	Right Part
$E \mapsto F \in S \times T$	$E \in S \;\wedge\; F \in T$
$S \in \mathbb{P}(T)$	$\forall x \cdot (x \in S \Rightarrow x \in T)$ (x is not free in S and T)
$E \in \{x \mid P\}$	$[x := E]P$ (x is not free in E)

Set Inclusion and Extensionality Axiom

Left Part	Right Part
$S \subseteq T$	$S \in \mathbb{P}(T)$
$S = T$	$S \subseteq T \;\wedge\; T \subseteq S$

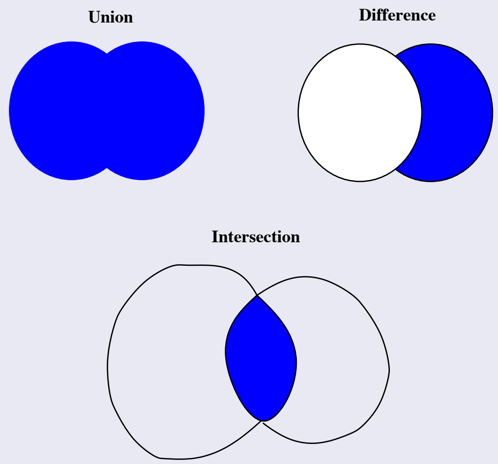
The first rule is just a syntactic extension

The second rule is the Extensionality Axiom

Elementary Set Operators

Union	$S \cup T$
Intersection	$S \cap T$
Difference	$S \setminus T$
Extension	$\{a, \ldots, b\}$
Empty set	$\varnothing$

Union, Difference, Intersection



Elementary Set Operator Memberships

$E \in S \cup T$	$E \in S \;\vee\; E \in T$
$E \in S \cap T$	$E \in S \;\wedge\; E \in T$
$E \in S \setminus T$	$E \in S \;\wedge\; E \notin T$
$E \in \{a, \ldots, b\}$	$E = a \;\vee\; \ldots \;\vee\; E = b$
$E \in \varnothing$	$\perp$

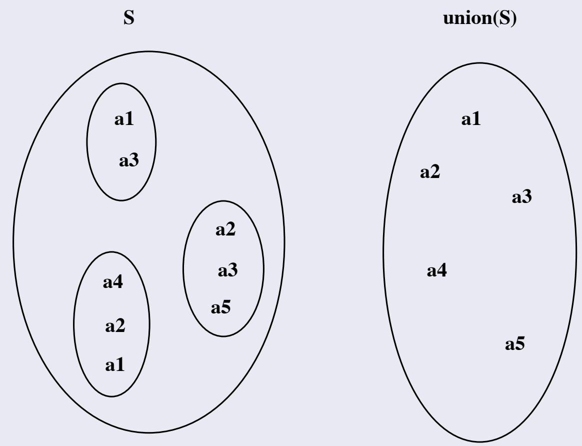
Summary of Basic and Elementary Operators

$S \times T$	$S \cup T$
$\mathbb{P}(S)$	$S \cap T$
	$S \setminus T$
$S \subseteq T$	$\{a, \ldots, b\}$
$S = T$	$\varnothing$

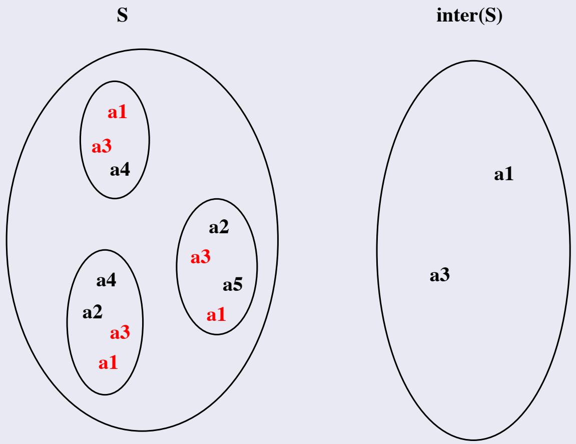
Generalizations of Elementary Operators

Generalized Union	$\mathrm{union}\,(S)$
Union Quantifier	$\bigcup x \cdot (P \mid T)$
Generalized Intersection	$\mathrm{inter}\,(S)$
Intersection Quantifier	$\bigcap x \cdot (P \mid T)$

Generalized Union



Generalized Intersection



Foundation for Deductive and Formal Proofs
A Quick Review of Propositional Calculus
A Quick Review of First Order Predicate Calculus
A Refresher on Set Theory
Basic Constructs
Extensions

Generalizations of Elementary Operator Memberships

$E \in \text{union}(S)$	$\exists s \cdot s \in S \wedge E \in s$ (s is not free in S and E)
$E \in (\bigcup x \cdot P \mid T)$	$\exists x \cdot P \wedge E \in T$ (x is not free in E)
$E \in \text{inter}(S)$	$\forall s \cdot s \in S \Rightarrow E \in s$ (s is not free in S and E)
$E \in (\bigcap x \cdot P \mid T)$	$\forall x \cdot P \Rightarrow E \in T$ (x is not free in E)

Well-definedness condition for case 3: $S \neq \varnothing$
Well-definedness condition for case 4: $\exists x \cdot P$

Foundation for Deductive and Formal Proofs
A Quick Review of Propositional Calculus
A Quick Review of First Order Predicate Calculus
A Refresher on Set Theory
Basic Constructs
Extensions

Summary of Generalizations of Elementary Operators

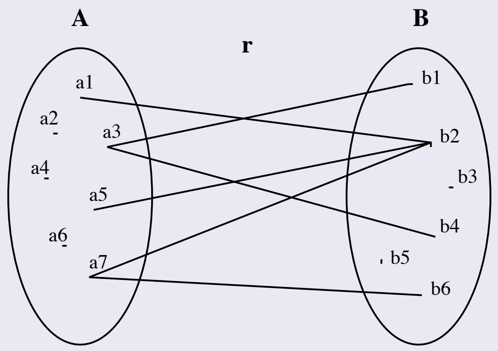
$\text{union}(S)$
$\bigcup x \cdot P \mid T$
$\text{inter}(S)$
$\bigcap x \cdot P \mid T$

Foundation for Deductive and Formal Proofs
A Quick Review of Propositional Calculus
A Quick Review of First Order Predicate Calculus
A Refresher on Set Theory
Basic Constructs
Extensions

Binary Relation Operators (1)

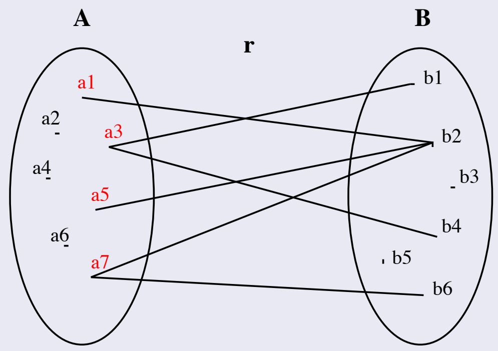
Binary relations	$S \leftrightarrow T$
Domain	$\text{dom}(r)$
Range	$\text{ran}(r)$
Converse	r^{-1}

A Binary Relation r from a Set A to a Set B



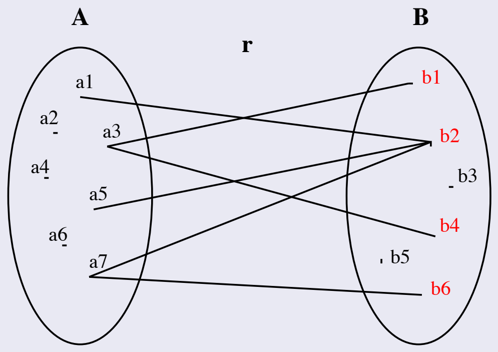
$$r \in A \leftrightarrow B$$

Domain of Binary Relation r



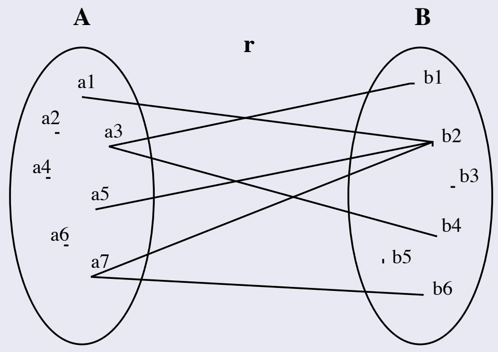
$$\mathrm{dom}(r) = \{a1, a3, a5, a7\}$$

Range of Binary Relation r



$$\mathrm{ran}(r) = \{b1, b2, b4, b6\}$$

Converse of Binary Relation r

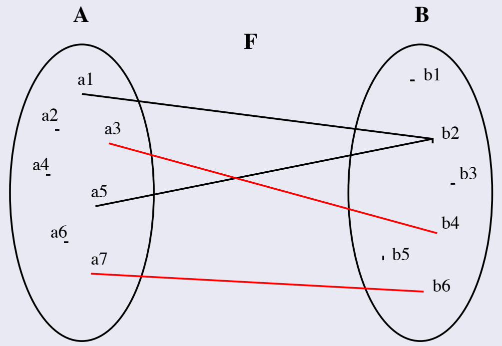


$$r^{-1} = \{b1 \mapsto a3, b2 \mapsto a1, b2 \mapsto a5, b2 \mapsto a7, b4 \mapsto a3, b6 \mapsto a7\}$$

Binary Relation Operators (3)

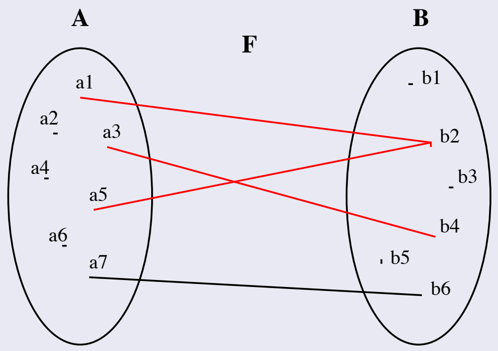
Domain restriction	$S \lhd r$
Range restriction	$r \rhd T$
Domain subtraction	$S \mathbin{⩤} r$
Range subtraction	$r \mathbin{⩥} T$

The Domain Restriction Operator



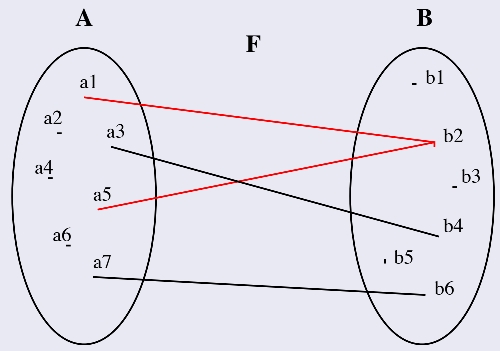
$\{a3,\ a7\} \lhd F$

The Range Restriction Operator



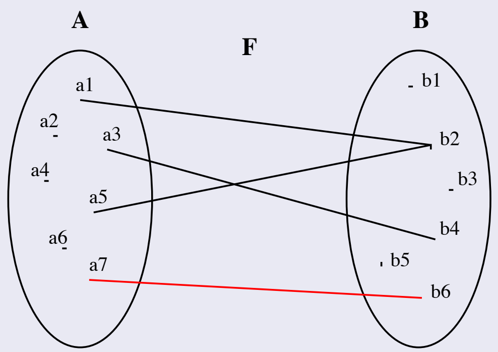
$F \rhd \{b2, b4\}$

The Domain Substraction Operator



$\{a3,\ a7\} ⩤ F$

The Range Substraction Operator



$F ⩥ \{b2, b4\}$

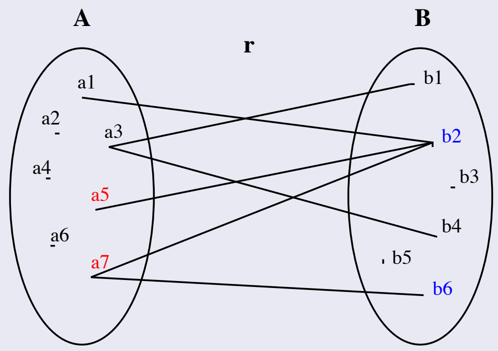
Binary Relation Operator Memberships (3)

Left Part	Right Part
$E \mapsto F \ \in \ S \lhd r$	$E \in S \ \land \ E \mapsto F \in r$
$E \mapsto F \ \in \ r \rhd T$	$E \mapsto F \in r \ \land \ F \in T$
$E \mapsto F \ \in \ S ⩤ r$	$E \notin S \ \land \ E \mapsto F \in r$
$E \mapsto F \ \in \ r ⩥ T$	$E \mapsto F \in r \ \land \ F \notin T$

Binary Relation Operators (4)

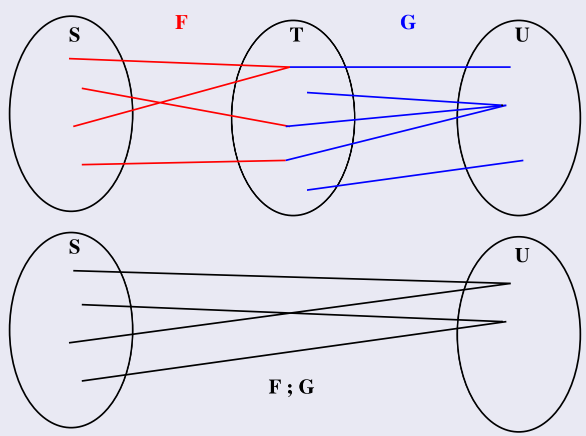
Image	$r[w]$
Composition	$p \; ; \; q$
Overriding	$p \lhd\!\!- q$
Identity	$\mathrm{id}\,(S)$

Image of $\{a5, a7\}$ under r

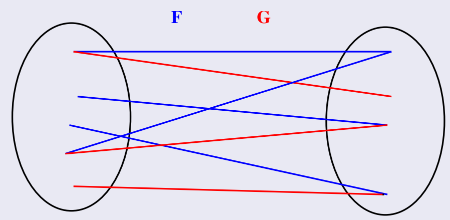


$$r[\{a5, a7\}] \;=\; \{b2, b6\}$$

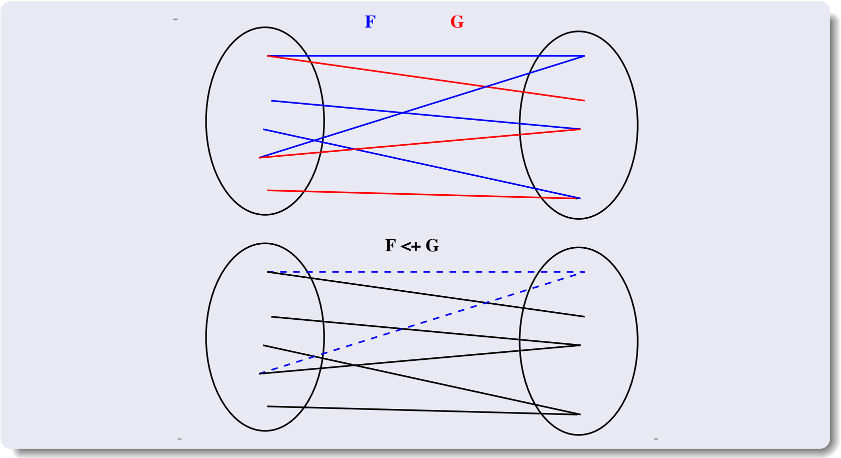
Forward Composition



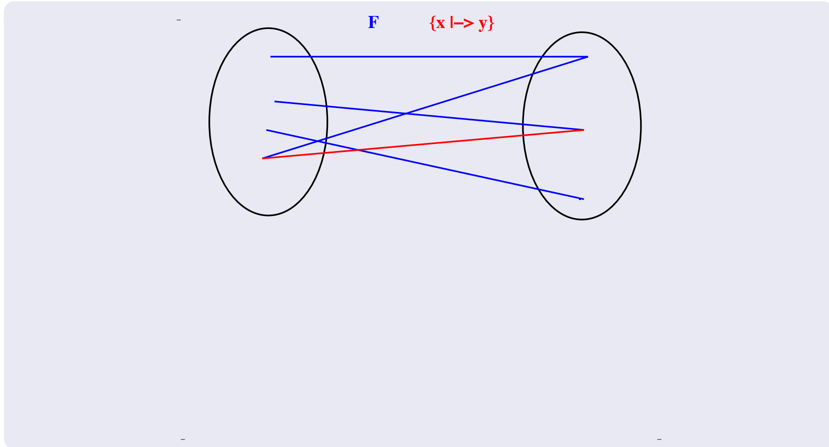
The Overriding Operator



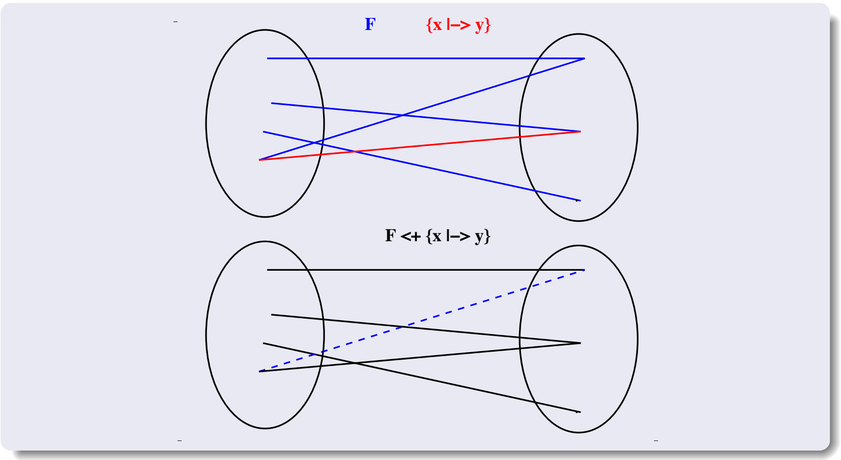
The Overriding Operator



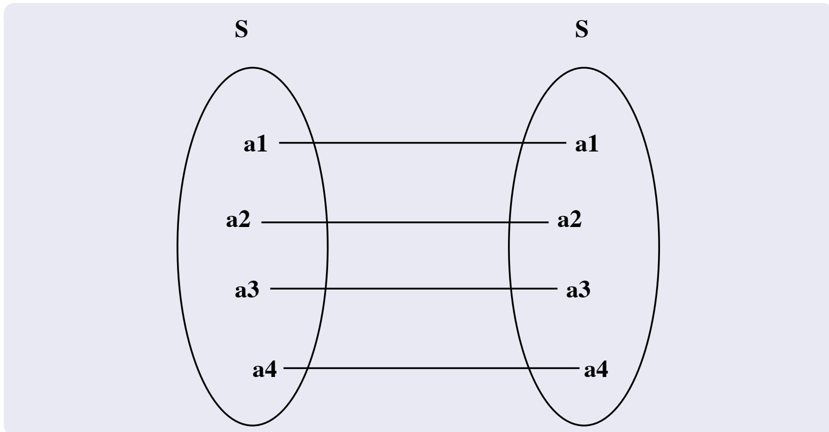
Special Case



Special Case



The Identity Relation



Classical Results with Relation Operators

$$r^{-1-1} = r$$

$$\mathrm{dom}(r^{-1}) = \mathrm{ran}(r)$$

$$(S \lhd r)^{-1} = r^{-1} \rhd S$$

$$(p\,;q)^{-1} = q^{-1}\,;p^{-1}$$

$$(p\,;q)\,;r = q\,;(p\,;r)$$

$$(p\,;q)[w] = q[p[w]]$$

$$p\,;(q \cup r) = (p\,;q) \cup (p\,;r)$$

$$r[a \cup b] = r[a] \cup r[b]$$

More classical Results

Given a relation r such that $r \in S \leftrightarrow S$

$r = r^{-1}$	r is symmetric
$r \cap r^{-1} = \varnothing$	r is asymmetric
$r \cap r^{-1} \subseteq \mathrm{id}(S)$	r is antisymmetric
$\mathrm{id}(S) \subseteq r$	r is reflexive
$r \cap \mathrm{id}(S) = \varnothing$	r is irreflexive
$r;r \subseteq r$	r is transitive

Translations into First Order Predicates

Given a relation r such that $r \in S \leftrightarrow S$

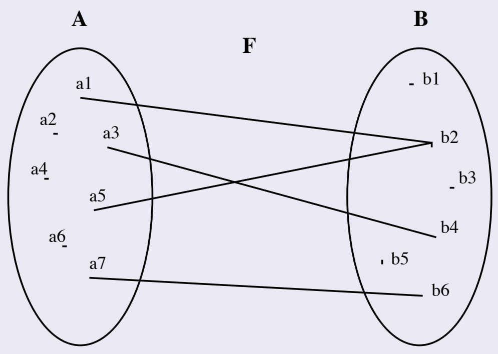
$r = r^{-1}$	$\forall x, y \cdot x \in S \land y \in S \Rightarrow (x \mapsto y \in r \Leftrightarrow y \mapsto x \in r)$
$r \cap r^{-1} = \varnothing$	$\forall x, y \cdot x \mapsto y \in r \Rightarrow y \mapsto x \notin r$
$r \cap r^{-1} \subseteq \mathrm{id}(S)$	$\forall x, y \cdot x \mapsto y \in r \land y \mapsto x \in r \Rightarrow x = y$
$\mathrm{id}(S) \subseteq r$	$\forall x \cdot x \in S \Rightarrow x \mapsto x \in r$
$r \cap \mathrm{id}(S) = \varnothing$	$\forall x, y \cdot x \mapsto y \in r \Rightarrow x \neq y$
$r;r \subseteq r$	$\forall x, y, z \cdot x \mapsto y \in r \land y \mapsto z \in r \Rightarrow x \mapsto z \in r$

Set-theoretic statements are far more readable than predicate calculus statements

Function Operators (1)

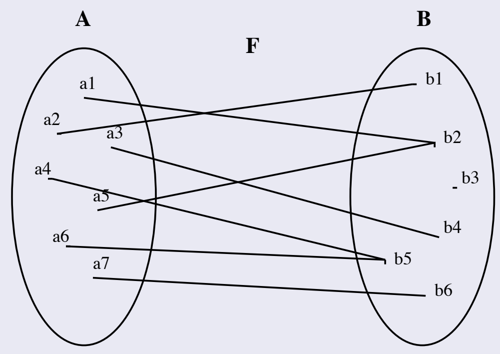
Partial functions	$S \pfun T$
Total functions	$S \rightarrow T$
Partial injections	$S \pinj T$
Total injections	$S \rightarrowtail T$

A Partial Function F from a Set A to a Set B



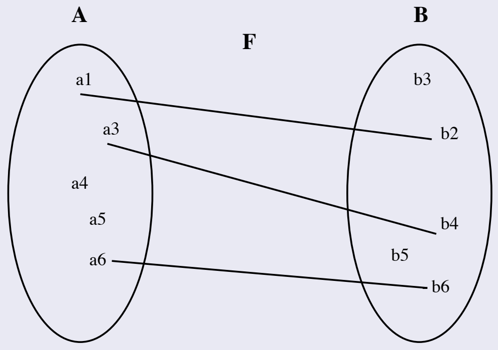
$F \in A \mathrel{+\!\!\!\!\rightarrow} B$

A Total Function F from a Set A to a Set B



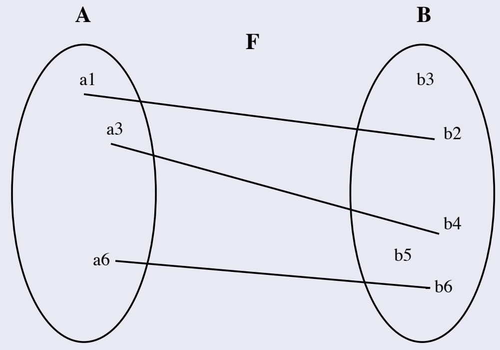
$F \in A \rightarrow B$

A Partial Injection F from a Set A to a Set B



$F \in A \mathrel{\rightarrowtail\!\!\!\!\!\!+\;} B$

A Total Injection F from a Set A to a Set B



$F \in A \rightarrowtail B$

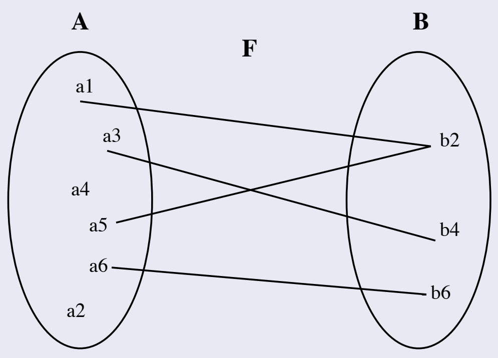
Function Operator Memberships (1)

Left Part	Right Part
$f \in S \mathrel{+\!\!\!\!\rightarrow} T$	$f \in S \leftrightarrow T \;\wedge\; (f^{-1}\,;f) = \mathrm{id}(\mathrm{ran}(f))$
$f \in S \rightarrow T$	$f \in S \mathrel{+\!\!\!\!\rightarrow} T \;\wedge\; s = \mathrm{dom}(f)$
$f \in S \mathrel{\rightarrowtail\!\!\!\!\!\!+} T$	$f \in S \mathrel{+\!\!\!\!\rightarrow} T \;\wedge\; f^{-1} \in T \mathrel{+\!\!\!\!\rightarrow} S$
$f \in S \rightarrowtail T$	$f \in S \rightarrow T \;\wedge\; f^{-1} \in T \mathrel{+\!\!\!\!\rightarrow} S$

Function Operators (2)

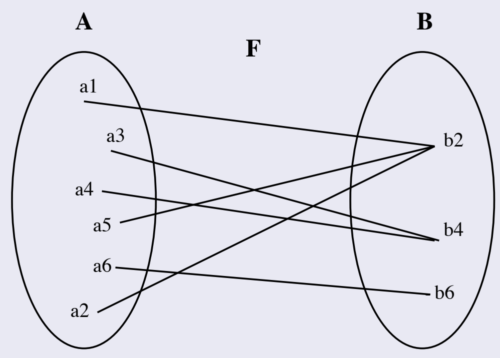
Partial surjections	$S \mathrel{+\!\!\!\!\twoheadrightarrow} T$
Total surjections	$S \twoheadrightarrow T$
Bijections	$S \mathrel{\rightarrowtail\!\!\!\!\!\twoheadrightarrow} T$

A Partial Surjection F from a Set A to a Set B



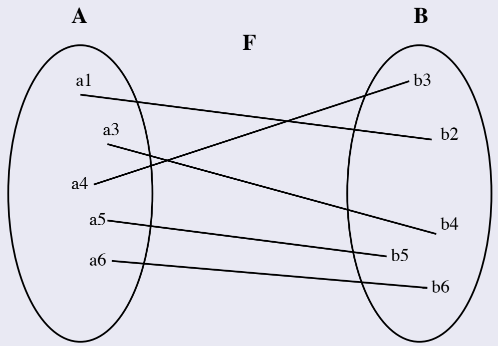
$F \in A \mathrel{+\!\!\!\!\twoheadrightarrow} B$

A Total Surjection F from a Set A to a Set B



$F \in A \twoheadrightarrow B$

A Bijection F from a Set A to a Set B



$F \in A ⤖ B$

Function Operator Memberships (2)

Left Part	Right Part
$f \in S ⤀ T$	$f \in S ⇸ T \;\wedge\; T = \mathrm{ran}(f)$
$f \in S \twoheadrightarrow T$	$f \in S \to T \;\wedge\; T = \mathrm{ran}(f)$
$f \in S ⤖ T$	$f \in S \rightarrowtail T \;\wedge\; f \in S \twoheadrightarrow T$

Summary of Function Operators

$S ⇸ T$	$S ⤀ T$
$S \to T$	$S \twoheadrightarrow T$
$S ⤔ T$	$S \rightarrowtail T$
$S ⤖ T$	

Summary of all Set-theoretic Operators (40)

$S \times T$	$S \setminus T$	r^{-1}	$r[w]$	$\mathrm{id}\,(S)$	$\{\,x \mid x \in S \;\wedge\; P\,\}$
$\mathbb{P}(S)$		$S \lhd r$ $S ⩤ r$	$p\,;q$	$S ⇸ T$ $S \to T$	
$S \subseteq T$		$r \rhd T$ $r ⩥ T$	$p \mathbin{<\!\!+} q$	$S ⤔ T$ $S \rightarrowtail T$	$\{\,a,\ b,\ \ldots,\ n\,\}$
$S \cup T$	$\mathrm{dom}\,(r)$ $\mathrm{ran}\,(r)$			$S ⤀ T$ $S \twoheadrightarrow T$	union $\bigcup$
$S \cap T$	$\varnothing$			$S ⤖ T$	inter $\bigcap$