Affine Extensions of Integer Vector Addition Systems with States

Michael Blondin

Joint work with Christoph Haase and Filip Mazowiecki
Vector addition systems with states (VASS)
Vector addition systems with states (VASS)

Control-states
Vector addition systems with states (VASS)

Control-states

Transitions

(0, 1, -1) (0, 0, 0) (0, -1, 2)

(1, 0, 0)
Vector addition systems with states (VASS)

(0, 1, -1)  (0, 0, 0)  (0, -1, 2)

p(0, 0, 1)

p(0, 0, 1)

p(1, 0, 2)

p(x, y, z)

0 < y + z ≤ 2

p(0, 0, 1)
Vector addition systems with states (VASS)

\[
\begin{align*}
(p(0,0,1), (0,1, -1)) & \quad (p(0,0,0), (0, 0, 0)) & \quad (p(0,0,1), (0, -1, 2)) \\
(p(1,0,2), (1, 0, 0)) & \quad (p(x, y, z), 0 < y + z < 2) \\
q(0, 0, 1) & \\
\end{align*}
\]
Vector addition systems with states (VASS)

\[
\begin{align*}
(0, 1, -1) & \quad (0, 0, 0) & \quad (0, -1, 2) \\
(1, 0, 0) & \\
\end{align*}
\]

\[
p(0, 1, 0)
\]

Control-states

Transitions

\[
p(0, 0, 1)
\]

\[
p(x, y, z)
\]

\[
0 < y + z \\ 2x
\]

1/11
Vector addition systems with states (VASS)

Control-states

Transitions

\[ p(0, 1, 0) \]

\[ \! N \]

\[ p(x, y, z) \]

\[ 0 < y + z \leq 2x \]

\[ p(0, 1, 0) \]

\[ q(0, -1, 2) \]

\[ (0, 1, -1) \]

\[ (0, 0, 0) \]

\[ (1, 0, 0) \]
Vector addition systems with states (VASS)

\[
\begin{align*}
(0, 1, -1) & \quad (0, 0, 0) & \quad (0, -1, 2) \\
p & \quad (1, 0, 0) & \quad q
\end{align*}
\]

\[p(0, 1, 0)\]
Vector addition systems with states (VASS)

\[ p(0, 1, -1) \rightarrow (0, 0, 0) \rightarrow (0, 1, 0) \]

\[ q(0, -1, 2) \rightarrow (0, 0, 0) \rightarrow (0, 1, 0) \]

\[ q(0, 1, 0) \]
Vector addition systems with states (VASS)

States:
- p: (0, 1, -1)
- q: (0, 0, 0)
- q(0, 1, 0)
- (0, -1, 2)

Transitions:
- p(0, 0, 1) \rightarrow q(0, 0, 0)
- q(0, 0, 0) \rightarrow p(1, 0, 2)
- q(0, 1, 0) \rightarrow q(0, 1, 0)

Control-states:
- p
- q

Constraint:
\[ x = y + z - 2 \]
Vector addition systems with states (VASS)

\[
\begin{align*}
\text{(0, 1, -1)} & \quad \text{(0, 0, 0)} & \quad \text{(0, -1, 2)} \\
\text{p} & \quad \text{q} \\
(1, 0, 0) & \\
\text{q(0, 0, 2)} & 
\end{align*}
\]
Vector addition systems with states (VASS)
Vector addition systems with states (VASS)

\[(0, 1, -1)\] \[\rightarrow\] \[(0, 0, 0)\] \[\rightarrow\] \[(0, -1, 2)\]

\[p(0, 0, 1)\] \[\rightarrow\] \[p(1, 0, 2)\]

\[0 < y + z\]

\[p(1, 0, 0)\]
Vector addition systems with states (VASS)

\[ \begin{align*} 
(0, 1, -1) & \quad (0, 0, 0) & \quad (0, -1, 2) \\
(1, 0, 0) & \\
\end{align*} \]

\[ \begin{align*} 
\text{p}(0, 0, 1) & \quad \rightarrow_{\mathbb{N}} \quad \text{p}(1, 0, 2) \\
\end{align*} \]
Vector addition systems with states (VASS)

\[ p(0, 0, 1) \xrightarrow{N} p(x, y, z) \iff 0 < y + z \leq 2^x \]
Vector addition systems with states (VASS)

Reachability: \( p(u) \xrightarrow{\mathbb{N}} q(v) \) ?

Coverability: \( p(u) \xrightarrow{\mathbb{N}} q(\geq v) \) ?
Vector addition systems with states (VASS)

Reachability: \( p(u) \xrightarrow{N} q(v) \)?

Coverability: \( p(u) \xrightarrow{N} q(\geq v) \)?
Vector addition systems with states (VASS)

Concurrent programs
Protocols
Business processes
Biological processes

Reachability: \[ p(u) \xrightarrow{\mathbb{N}} q(v) \]?

Coverability: \[ p(u) \xrightarrow{\mathbb{N}} q(\geq v) \]?

Correct?
Vector addition systems with states (VASS)

Common operations used for modeling:

- **Reset**
  - $x \leftarrow 0$

- **Swap**
  - $x \leftrightarrow y$

- **Transfer**
  - $x \leftarrow x + y$
  - $y \leftarrow 0$

- **Copy**
  - $x \leftarrow y$
Vector addition systems with states (VASS)

Common operations used for modeling:

- **Reset**
  \[
  \begin{pmatrix}
  0 & 0 \\
  0 & 1 \\
  \end{pmatrix}
  \cdot
  \begin{pmatrix}
  x \\
  y \\
  \end{pmatrix}
  \rightarrow
  \begin{pmatrix}
  0 & 0 \\
  0 & 1 \\
  \end{pmatrix}
  \cdot
  \begin{pmatrix}
  x \\
  y \\
  \end{pmatrix}
  \]

- **Swap**
  \[
  \begin{pmatrix}
  0 & 1 \\
  1 & 0 \\
  \end{pmatrix}
  \cdot
  \begin{pmatrix}
  x \\
  y \\
  \end{pmatrix}
  \rightarrow
  \begin{pmatrix}
  0 & 1 \\
  1 & 0 \\
  \end{pmatrix}
  \cdot
  \begin{pmatrix}
  x \\
  y \\
  \end{pmatrix}
  \]

- **Transfer**
  \[
  \begin{pmatrix}
  1 & 1 \\
  0 & 0 \\
  \end{pmatrix}
  \cdot
  \begin{pmatrix}
  x \\
  y \\
  \end{pmatrix}
  \rightarrow
  \begin{pmatrix}
  1 & 1 \\
  0 & 0 \\
  \end{pmatrix}
  \cdot
  \begin{pmatrix}
  x \\
  y \\
  \end{pmatrix}
  \]

- **Copy**
  \[
  \begin{pmatrix}
  0 & 1 \\
  0 & 1 \\
  \end{pmatrix}
  \cdot
  \begin{pmatrix}
  x \\
  y \\
  \end{pmatrix}
  \rightarrow
  \begin{pmatrix}
  0 & 1 \\
  0 & 1 \\
  \end{pmatrix}
  \cdot
  \begin{pmatrix}
  x \\
  y \\
  \end{pmatrix}
  \]

All affine transformations!
Affine VASS:

\[ A \cdot x + b \in \mathbb{Z}^n \]

\[ \exists \mathbb{Z}^{n \times n} \]

\[ \in \mathbb{Z}^n \]
## Complexity of reachability and coverability

<table>
<thead>
<tr>
<th></th>
<th>No extensions</th>
<th>+ Resets</th>
<th>+ Transfers</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \star \to_{\mathbb{N}} )</td>
<td>TOWER-hard ((\text{CLLLM '19}))</td>
<td>Undecidable ((\text{Araki, Kasami '76}))</td>
<td>Ackermann-complete ((\text{Schnoebelen '02, Figueira et al. '11}))</td>
</tr>
<tr>
<td>( \star \to_{\mathbb{N}} \geq )</td>
<td>EXPSPACE-complete ((\text{Lipton '76, Rackoff '78}))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Complexity of reachability and coverability

<table>
<thead>
<tr>
<th></th>
<th>No extensions</th>
<th>+ Resets</th>
<th>+ Transfers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$*\rightarrow\mathbb{N}$</td>
<td>TOWER-hard <em>(CLLM '19)</em> [∈] Ackermann <em>(Leroux, Schmitz '19)</em></td>
<td>Undecidable <em>(Araki, Kasami '76)</em></td>
<td>Ackermann-complete <em>(Schnoebelen '02, Figueira et al. '11)</em></td>
</tr>
<tr>
<td>$\not{*}\rightarrow\mathbb{N}$</td>
<td>EXPSPACE-complete <em>(Lipton '76, Rackoff '78)</em></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Intractable!*
## Complexity of reachability and coverability

<table>
<thead>
<tr>
<th>No extensions</th>
<th>+ Resets</th>
<th>+ Transfers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rightarrow \mathbb{N}$ TOWER-hard (CLLLM ’19)</td>
<td>Undecidable (Araki, Kasami ’76)</td>
<td>Ackermann-complete</td>
</tr>
<tr>
<td>$\in$ Ackermann (Leroux, Schmitz ’19)</td>
<td></td>
<td>(Schnoebelen ’02, Figueira et al. ’11)</td>
</tr>
<tr>
<td>$\rightarrow \mathbb{N} \geq$ EXPSPACE-complete</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Lipton ’76, Rackoff ’78)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Can be alleviated by using an over-approximation of $\rightarrow \mathbb{N}$
- Successful in practice, *e.g.* Esparza *et al.* CAV’14, B. *et al.* TACAS’16, Geffroy *et al.* RP’16, Athanasiou *et al.* IJCAR’16
- We consider $\mathbb{Z}$-VASS: counters allowed to drop below 0
### Complexity of reachability and coverability

<table>
<thead>
<tr>
<th></th>
<th>No extensions</th>
<th>+ Resets</th>
<th>+ Transfers</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rightarrow_\mathbb{N} )</td>
<td>TOWER-hard ((\text{CLLLM '19})) (\subseteq \text{Ackermann} \ (\text{Leroux, Schmitz '19}))</td>
<td>Undecidable ((\text{Araki, Kasami '76}))</td>
<td>Ackermann-complete ((\text{Schnoebelen '02, Figueira et al. '11}))</td>
</tr>
<tr>
<td>( \rightarrow_\mathbb{N} \geq )</td>
<td>EXPSPACE-complete ((\text{Lipton '76, Rackoff '78}))</td>
<td>(</td>
<td>)</td>
</tr>
</tbody>
</table>

- Can be alleviated by using an over-approximation of \( \rightarrow_\mathbb{N} \)
- Successful in practice, \(\text{e.g. Esparza et al. CAV'14, B. et al. TACAS'16,}\)
  \(\text{Geffroy et al. RP'16, Athanasiou et al. IJCAR'16}\)
- We consider \(\mathbb{Z}\)-VASS: counters allowed to drop below 0
<table>
<thead>
<tr>
<th>Complexity of reachability and coverability</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No extensions</strong></td>
</tr>
<tr>
<td>$* \rightarrow \mathbb{N}$</td>
</tr>
<tr>
<td>TOWER-hard ((\text{CLLLM '19}))</td>
</tr>
<tr>
<td>$\in$ Ackermann ((\text{Leroux, Schmitz '19}))</td>
</tr>
<tr>
<td>+ Resets</td>
</tr>
<tr>
<td>$* \rightarrow \mathbb{N}$</td>
</tr>
<tr>
<td>EXPSPACE-complete ((\text{Lipton '76, Rackoff '78}))</td>
</tr>
<tr>
<td>+ Transfers</td>
</tr>
<tr>
<td>$* \rightarrow \mathbb{Z}$</td>
</tr>
<tr>
<td>NP-complete ((\text{Haase, Halfon '14}))</td>
</tr>
<tr>
<td>$* \rightarrow \mathbb{Z}$</td>
</tr>
<tr>
<td>$? \geq$</td>
</tr>
</tbody>
</table>

\( \text{NP-complete} \) \((\text{Haase, Halfon '14})\)
## Complexity of reachability and coverability

<table>
<thead>
<tr>
<th></th>
<th>No extensions</th>
<th>+ Resets</th>
<th>+ Transfers</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rightarrow \mathbb{N} )</td>
<td>TOWER-hard (CLLLM '19) [\in] Ackermann (Leroux, Schmitz '19)</td>
<td>Undecidable (Araki, Kasami '76)</td>
<td></td>
</tr>
<tr>
<td>( \rightarrow \mathbb{N} \geq )</td>
<td>EXPSPACE-complete (Lipton '76, Rackoff '78)</td>
<td>Ackermann-complete (Schnoebelen '02, Figueira et al. '11)</td>
<td></td>
</tr>
<tr>
<td>( \rightarrow \mathbb{Z} )</td>
<td>NP-complete (new proof)</td>
<td></td>
<td>PSPACE-complete</td>
</tr>
<tr>
<td>( \rightarrow \mathbb{Z} \geq )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Our contribution

- Any affine \( \mathbb{Z} \)-VASS with finite matrix monoid can be translated into an equivalent \( \mathbb{Z} \)-VASS

- Reachability relation of such affine \( \mathbb{Z} \)-VASS is semilinear

- Classification of complexity w.r.t. extensions
Complexity of reachability and coverability

Related work

- **Finkel and Leroux (FSTTCS’12)**
  Accelerations of affine counter machines without control-states

- **Iosif and Sangnier (ATVA’16)**
  Complexity of model checking over flat structures with guards defined by convex polyhedra

- **Cadilhac, Finkel and McKenzie (IJFCS’12)**
  Affine Parikh automata with finite-monoid restriction
Overview

Infinite monoids

Affine \( \mathbb{Z} \)-VASS

Transfer + copy \( \mathbb{Z} \)-VASS

Transfer \( \mathbb{Z} \)-VASS

Copy \( \mathbb{Z} \)-VASS

Reset + permutation \( \mathbb{Z} \)-VASS

Finite monoids

Reset \( \mathbb{Z} \)-VASS

Permutation \( \mathbb{Z} \)-VASS

\( \mathbb{Z} \)-VASS

Remarks on monoids over \( \mathbb{N} \)

Finite monoids

Decidable

Infinite monoids

Undecidable

NP-complete

PSPACE-complete
Overview

Infinite monoids

- Affine $\mathbb{Z}$-VASS
- Transfer + copy $\mathbb{Z}$-VASS

Finite monoids

- Transfer $\mathbb{Z}$-VASS
- Copy $\mathbb{Z}$-VASS
- Reset + permutation $\mathbb{Z}$-VASS
- Reset $\mathbb{Z}$-VASS
- Permutation $\mathbb{Z}$-VASS
- $\mathbb{Z}$-VASS

Remarks on monoids over $\mathbb{N}$

Decidable

Infinite monoids

- Undecidable

Finite monoids

- Decidable
Overview

Infinite monoids

Affine $\mathbb{Z}$-VASS

Transfer + copy $\mathbb{Z}$-VASS

Transfer $\mathbb{Z}$-VASS

Copy $\mathbb{Z}$-VASS

Reset + permutation $\mathbb{Z}$-VASS

Finite monoids

Reset $\mathbb{Z}$-VASS

Permutation $\mathbb{Z}$-VASS

$\mathbb{Z}$-VASS

NP-complete

Decidable
Overview

Infinite monoids

Affine $\mathbb{Z}$-VASS

Transfer + copy $\mathbb{Z}$-VASS

Transfer $\mathbb{Z}$-VASS

Copy $\mathbb{Z}$-VASS

Reset + permutation $\mathbb{Z}$-VASS

PSPACE-complete

Finite monoids

NP-complete

Reset $\mathbb{Z}$-VASS

Permutation $\mathbb{Z}$-VASS

Decidable
Overview

1. Decidable (+ semilinear)
   - Reset \( \mathbb{Z} \)-VASS
   - Permutation \( \mathbb{Z} \)-VASS

2. PSPACE-complete
   - Transfer \( \mathbb{Z} \)-VASS
   - Copy \( \mathbb{Z} \)-VASS
   - Reset + permutation \( \mathbb{Z} \)-VASS

3. Undecidable
   - Affine \( \mathbb{Z} \)-VASS
   - Transfer + copy \( \mathbb{Z} \)-VASS

4. Remarks on monoids over \( \mathbb{N} \)
   - Decidable (+ semilinear)
     - Infinite monoids
     - Finite monoids
For every transition $t: \mathcal{P} \xrightarrow{A\cdot x + b} \mathcal{Q}$ and $\sigma \in T^*$, let

$$M_\varepsilon = I \quad \varepsilon(u) = u$$

$$M_{\sigma t} = A \cdot M_\sigma \quad \sigma t(u) = A \cdot \sigma(u) + b$$
For every transition $t$: $p \xrightarrow{A \cdot x + b} q$ and $\sigma \in T^*$, let

$$M_{\varepsilon} = I$$

$$M_{\sigma t} = A \cdot M_{\sigma}$$

$$\varepsilon(u) = u$$

$$\sigma t(u) = A \cdot \sigma(u) + b$$

Matrix
A few definitions

For every transition $t$: $p \xrightarrow{A \cdot x + b} q$ and $\sigma \in T^*$, let

\[
M_{\varepsilon} = I \\
M_{\sigma t} = A \cdot M_{\sigma} \\
\varepsilon(u) = u \\
\sigma t(u) = A \cdot \sigma(u) + b
\]
A few definitions

For every transition $t$: $\bigcirc p \xrightarrow{A \cdot x + b} \bigcirc q$ and $\sigma \in T^*$, let

\[
\begin{align*}
M_\varepsilon &= I & \varepsilon(u) &= u \\
M_{\sigma^t} &= A \cdot M_\sigma & \sigma^t(u) &= A \cdot \sigma(u) + b
\end{align*}
\]

Matrix monoid

\[
\mathcal{M}_V = \{M_w : w \in T^*\}
\]
### Theorem

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $\mathcal{M}_\mathcal{V}$ is finite, then $\exists \mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(u) \rightarrow_{\mathbb{Z}} q(v)$ in $\mathcal{V} \iff p(u,0) \rightarrow_{\mathbb{Z}} q(0,v)$ in $\mathcal{V}'$

- $|\mathcal{V}'| \in \text{poly}(|\mathcal{V}|, |\mathcal{M}_\mathcal{V}|)$
## From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

### Theorem

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $\mathcal{M}_\mathcal{V}$ is finite, then $\exists \mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(u) \overset{*}{\rightarrow}_\mathbb{Z} q(v)$ in $\mathcal{V} \iff p(u, 0) \overset{*}{\rightarrow}_\mathbb{Z} q(0, v)$ in $\mathcal{V}'$
- $|\mathcal{V}'| \in \text{poly}(|\mathcal{V}|, |\mathcal{M}_\mathcal{V}|)$

### Proof sketch

$p(u) \overset{w}{\rightarrow}_\mathbb{Z} q(v) \iff$

- $w$ is a path from $p$ to $q$
- $v = w(u)$
**Theorem**

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $M_{\mathcal{V}}$ is finite, then $\exists \mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(u) \xrightarrow{Z} q(v)$ in $\mathcal{V}$ $\iff$ $p(u, 0) \xrightarrow{Z} q(0, v)$ in $\mathcal{V}'$

- $|\mathcal{V}'| \in \text{poly}(|\mathcal{V}|, |M_{\mathcal{V}}|)$

**Proof sketch**

$p(u) \xrightarrow{w} q(v)$ $\iff$ $w$ is a path from $p$ to $q$

- $v = M_w \cdot u + w(0)$
From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

**Theorem**

Let $V$ be an affine $\mathbb{Z}$-VASS. If $M_V$ is finite, then $\exists$ $\mathbb{Z}$-VASS $V'$ s.t.

- $p(u) \rightarrow_{\mathbb{Z}} q(v)$ in $V$ $\iff$ $p(u, 0) \rightarrow_{\mathbb{Z}} q(0, v)$ in $V'$
- $|V'| \in \text{poly}(|V|, |M_V|)$

**Proof sketch**

$p(u) \rightarrow_{\mathbb{Z}} q(v) \iff$

- $w$ is a path from $p$ to $q$
- $v = M_w \cdot u + w(0)$
Theorem

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $\mathcal{M}_\mathcal{V}$ is finite, then $\exists$ $\mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(u) \overset{*}{\rightarrow}_\mathbb{Z} q(v)$ in $\mathcal{V} \iff p(u, 0) \overset{*}{\rightarrow}_\mathbb{Z} q(0, v)$ in $\mathcal{V}'$
- $|\mathcal{V}'| \in \text{poly}(|\mathcal{V}|, |\mathcal{M}_\mathcal{V}|)$

Proof sketch

$p(u) \overset{w}{\rightarrow}_\mathbb{Z} q(v) \iff \cdot w$ is a path from $p$ to $q$

- $v = M_w \cdot u + w(0)$
From affine \( \mathbb{Z} \)-VASS to \( \mathbb{Z} \)-VASS

**Theorem**

Let \( \mathcal{V} \) be an affine \( \mathbb{Z} \)-VASS. If \( \mathcal{M}_\mathcal{V} \) is finite, then \( \exists \mathbb{Z} \)-VASS \( \mathcal{V}' \) s.t.

\( p(u) \xrightarrow{\mathcal{Z}} q(v) \) in \( \mathcal{V} \) \iff \( p(u, 0) \xrightarrow{\mathcal{Z}} q(0, v) \) in \( \mathcal{V}' \)

\( |\mathcal{V}'| \in \text{poly}(|\mathcal{V}|, |\mathcal{M}_\mathcal{V}|) \)

**Proof sketch**

\( p(u) \xrightarrow{\mathcal{Z}} q(v) \iff \cdot w \text{ is a path from } p \text{ to } q \)

\( \cdot v = M_w \cdot u + w(0) \)

Guess \( M_w \)

Compute \( w(0) \)
From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

**Theorem**

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $\mathcal{M}_\mathcal{V}$ is finite, then $\exists$ $\mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(u) \xrightarrow{Z} q(v)$ in $\mathcal{V} \iff p(u, 0) \xrightarrow{Z} q(0, v)$ in $\mathcal{V}'$
- $|\mathcal{V}'| \in \text{poly}(|\mathcal{V}|, |\mathcal{M}_\mathcal{V}|)$

**Proof sketch**

$p(u) \xrightarrow{Z} q(v) \iff \cdot w$ is a path from $p$ to $q$

\[ v = M_w \cdot u + w(0) \]
From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

**Theorem**

Let $V$ be an affine $\mathbb{Z}$-VASS. If $\mathcal{M}_V$ is finite, then $\exists \mathbb{Z}$-VASS $V'$ s.t.

- $p(u) \rightarrow^*_\mathbb{Z} q(v)$ in $V \iff p(u, 0) \rightarrow^*_\mathbb{Z} q(0, v)$ in $V'$
- $|V'| \in \text{poly}(|V|, |\mathcal{M}_V|)$

**Proof sketch**

$p(u) \rightarrow^w_\mathbb{Z} q(v) \iff \cdot w$ is a path from $p$ to $q$

- $v = M_w \cdot u + w(0)$
From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

**Theorem**

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $\mathcal{M}_\mathcal{V}$ is finite, then $\exists \mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(u) \xrightarrow{\mathcal{V}} q(v)$ in $\mathcal{V} \iff p(u, 0) \xrightarrow{\mathcal{V}} q(0, v)$ in $\mathcal{V}'$
- $|\mathcal{V}'| \in \text{poly}(|\mathcal{V}|, |\mathcal{M}_\mathcal{V}|)$

**Proof sketch**

$p(u) \xrightarrow{\mathcal{V}} q(v) \iff$ 

- $w$ is a path from $p$ to $q$
- $v = M_w \cdot u + w(0)$

Guess $M_w$  

$u, 0$  

Guess end

$u, w(0)$  

$0, M_w \cdot u + w(0)$
From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

**Theorem**

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $M_{\mathcal{V}}$ is finite, then $\exists$ $\mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(u) \Rightarrow_{\mathbb{Z}} q(v)$ in $\mathcal{V}$ $\iff$ $p(u, 0) \Rightarrow_{\mathbb{Z}} q(0, v)$ in $\mathcal{V}'$

- $|\mathcal{V}'| \in \text{poly}(|\mathcal{V}|, |M_{\mathcal{V}}|)$

**Proof sketch**

$p(u) \xrightarrow{w} \mathbb{Z} q(v)$ $\iff$ $w$ is a path from $p$ to $q$

- $v = M_w \cdot u + w(0)$

```
Guess $M_w$
Compute $w(0)$
Compute $M_w \cdot u$
Guess end
```

$p$ $\rightarrow$ $p, M_w, M_w$ $\rightarrow$ $q, I, M_w$ $\rightarrow$ $q, I, M_w$ $\rightarrow$ $q$

$u, 0$ $\rightarrow$ $u, 0$ $\rightarrow$ $u, w(0)$ $\rightarrow$ $0, M_w \cdot u + w(0)$ $\rightarrow$ $0, M_w \cdot u + w(0)$
Theorem

Let \( \mathcal{V} \) be an affine \( \mathbb{Z} \)-VASS. If \( \mathcal{M}_\mathcal{V} \) is finite, then \( \exists \mathbb{Z} \)-VASS \( \mathcal{V}' \) s.t.

- \( p(u) \xrightarrow{\mathbb{Z}} q(v) \) in \( \mathcal{V} \iff p(u, 0) \xrightarrow{\mathbb{Z}} q(0, v) \) in \( \mathcal{V}' \)
- \( |\mathcal{V}'| \in \text{poly}(|\mathcal{V}|, |\mathcal{M}_\mathcal{V}|) \)

Corollary

Reachability is decidable for

affine \( \mathbb{Z} \)-VASS with finite matrix monoid
Corollary

If an affine $\mathbb{Z}$-VASS has a finite monoid, then

$$\left\{ (u, v) : p(u) \rightarrow^{*} \mathbb{Z} q(v) \right\} \text{ is semilinear}$$

Proof

Follows from our translation and known result on $\mathbb{Z}$-VASS (Haase, Halfon RP'14)
Corollary

If an affine $\mathbb{Z}$-VASS has a finite monoid, then

$$\{(u, v) : p(u) \to^*_{\mathbb{Z}} q(v)\}$$

is semilinear

Observation

Converse is not true:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot x$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot x$$

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
Corollary

If an affine \( \mathbb{Z} \)-VASS has a finite monoid, then

\[
\{(u, v) : p(u) \rightarrow^{\mathbb{Z}} q(v)\}
\]

is semilinear

Observation

Boigelot ’98, Finkel and Leroux ’02

Converse is true for single state and single transition:

\[
A \cdot x + b
\]
Reachability in transfer $\mathbb{Z}$-VASS is in PSPACE

- Transfer matrix: exactly one 1 per column, hence $|\mathcal{M}_\mathcal{V}| \leq n^n = 2^{n \log n}$

- Transform transfer $\mathbb{Z}$-VASS $\mathcal{V}$ into $\mathbb{Z}$-VASS $\mathcal{V}'$ of size $\text{poly}(|\mathcal{V}|, 2^{n \log n})$

- $\mathbb{Z}$-reachability has witnesses of the form $w_1^{k_1}w_2^{k_2} \cdots w_\ell^{k_\ell}$ where $|w_1w_2 \cdots w_\ell| \leq \text{poly}(|\mathcal{V}'|)$ (B. et al. LICS’15)

- Guess witness on the fly with polynomial space
Reachability in transfer $\mathbb{Z}$-VASS is in PSPACE

- Transfer matrix: exactly one 1 per column, hence $|\mathcal{M}_\mathcal{V}| \leq n^n = 2^{n \log n}$

- Transform transfer $\mathbb{Z}$-VASS $\mathcal{V}$ into $\mathbb{Z}$-VASS $\mathcal{V}'$
  of size $\text{poly}(|\mathcal{V}|, 2^{n \log n})$

- $\mathbb{Z}$-reachability has witnesses of the form $w_{1}^{k_1}w_{2}^{k_2} \cdots w_{\ell}^{k_\ell}$
  where $|w_1 w_2 \cdots w_\ell| \leq \text{poly}(|\mathcal{V}'|)$ (B. et al. LICS'15)

- Guess witness on the fly with polynomial space
Reachability in transfer $\mathbb{Z}$-VASS is in PSPACE

- Transfer matrix: exactly one 1 per column, hence $|\mathcal{M}_\mathcal{V}| \leq n^n = 2^{n \log n}$

- Transform transfer $\mathbb{Z}$-VASS $\mathcal{V}$ into $\mathbb{Z}$-VASS $\mathcal{V}'$ of size $\text{poly}(|\mathcal{V}|, 2^{n \log n})$

- $\mathbb{Z}$-reachability has witnesses of the form $w_1^{k_1}w_2^{k_2} \cdots w_\ell^{k_\ell}$ where $|w_1w_2 \cdots w_\ell| \leq \text{poly}(|\mathcal{V}'|)$ (B. et al. LICS’15)

- Guess witness on the fly with polynomial space
Reachability in transfer \( \mathbb{Z} \)-VASS is in PSPACE

- Transfer matrix: exactly one 1 per column, hence \( |M_V| \leq n^n = 2^{n\log n} \)

- Transform transfer \( \mathbb{Z} \)-VASS \( V \) into \( \mathbb{Z} \)-VASS \( V' \) of size \( \text{poly}(|V|, 2^{n\log n}) \)

- \( \mathbb{Z} \)-reachability has witnesses of the form \( w_1^{k_1}w_2^{k_2} \cdots w_\ell^{k_\ell} \) where \( |w_1w_2 \cdots w_\ell| \leq \text{poly}(|V'|) \) (B. et al. LICS‘15)

- Guess witness on the fly with polynomial space
Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

$q$

\[\begin{array}{cccc}
a & b & \ldots & b \\
\end{array}\]

Simulation is faithful iff the sum of bits is left unchanged

Swaps and resets can be simulated by transfers
Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

$q_0(\begin{array}{cccccc} 1 & 0 & 0 & 1 & \cdots & 0 & 1 \end{array})$
Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

Simulation is faithful iff the sum of bits is left unchanged

Swaps and resets can be simulated by transfers
Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

Simulation is faithful iff

Swaps and resets can be simulated by transfers

$r_1( 1 \ 0 \ 0 \ 0 \ 1 \ 
\ldots \ 
0 \ 1 )$
Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

\[
\begin{pmatrix}
 a & b & \cdots & b \\
 a & b & \cdots & b \\
 1 & 0 & 0 & 1 & \cdots & 0 & 1
\end{pmatrix}
\]

swap
Reachability in transfer \(\mathbb{Z}\)-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

\[
\begin{array}{ccccccc}
  & & & & & & \\
  & & & & & & \\
  & & r & & & & \\
  & & & & & & \\
  a & b & b & \cdots & b \\
  a & b & a & b & & & \\
  r_1 ( 0 & 1 & 0 & 1 & \cdots & 0 & 1 )
\end{array}
\]
Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

\[ r \]

\[
\begin{array}{cccccc}
\text{a} & \text{b} & \text{b} & \text{…} & \text{b} \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{a} & \text{b} \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & \text{…} & 0 & 1 \\
\end{array}
\]

Reset
Reachability in transfer \( \mathbb{Z} \)-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

Simulation is faithful iff the sum of bits is left unchanged
Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

Swaps and resets can be simulated by transfers
Reachability in affine \( \mathbb{Z} \)-VASS is undecidable

Reduction from the Post correspondence problem

\[
\begin{align*}
    w_1 &= \begin{array}{c} 10 \end{array} \\
        &= \begin{array}{c} 1 \end{array} \\
    w_2 &= \begin{array}{c} 01 \end{array} \\
        &= \begin{array}{c} 0 \end{array} \\
    w_3 &= \begin{array}{c} 1 \end{array} \\
        &= \begin{array}{c} 011 \end{array}
\end{align*}
\]
Reachability in affine $\mathbb{Z}$-VASS is undecidable

Reichert ’15

Reduction from the Post correspondence problem

\[ w_1 = \frac{10}{1} \quad w_2 = \frac{01}{0} \quad w_3 = \frac{1}{011} \]

\[ p \quad \xrightarrow{\text{x } \leftarrow 4\text{x } + 1 \quad \text{y } \leftarrow 2\text{y}} \quad x \leftarrow 4\text{x } + 2 \quad y \leftarrow 2\text{y} + 1 \]

\[ q \quad \xrightarrow{\text{x } \leftarrow \text{x } - 1 \quad \text{y } \leftarrow \text{y } - 1} \quad x \leftarrow \text{x } - 1 \quad y \leftarrow \text{y } - 1 \]

\[ x \leftarrow 2\text{x } + 1 \quad y \leftarrow 8\text{y } + 3 \]
Reachability in affine $\mathbb{Z}$-VASS is undecidable

Reichert ’15

Reduction from the Post correspondence problem

\[ w_1 = \begin{array}{c} 10 \\ 1 \end{array}, \quad w_2 = \begin{array}{c} 01 \\ 0 \end{array}, \quad w_3 = \begin{array}{c} 1 \\ 011 \end{array} \]

\[
\begin{align*}
x &\leftarrow 4x + 2 \\
y &\leftarrow 2y + 1
\end{align*}
\]

\[
\begin{align*}
x &\leftarrow 4x + 1 \\
y &\leftarrow 2y
\end{align*}
\]

\[
\begin{align*}
x &\leftarrow x - 1 \\
y &\leftarrow x - 1
\end{align*}
\]

\[
\begin{align*}
x &\leftarrow 2x + 1 \\
y &\leftarrow 8y + 3
\end{align*}
\]

\[
\begin{align*}
x &\leftarrow x - 1 \\
y &\leftarrow y - 1
\end{align*}
\]
Reachability in affine $\mathbb{Z}$-VASS is undecidable

Reichert '15

Reduction from the Post correspondence problem

$$w_1 = \frac{10}{1} \quad w_2 = \frac{01}{0} \quad w_3 = \frac{1}{011}$$

\[
\begin{align*}
x &\leftarrow 4x + 2 \\
y &\leftarrow 2y + 1
\end{align*}
\]

\[
\begin{align*}
x &\leftarrow x - 1 \\
y &\leftarrow y - 1
\end{align*}
\]
Reachability in affine $\mathbb{Z}$-VASS is undecidable

Reichert ‘15

Reduction from the Post correspondence problem

\[ w_1 = \begin{array}{c} 10 \\ 1 \end{array} \quad w_2 = \begin{array}{c} 01 \\ 0 \end{array} \quad w_3 = \begin{array}{c} 1 \\ 011 \end{array} \]

\[ \begin{align*}
  &x \leftarrow 4x + 2 \\
  &y \leftarrow 2y + 1 \\
  &x \leftarrow 4x + 1 \\
  &y \leftarrow 2y \\
  &x \leftarrow 2x + 1 \\
  &y \leftarrow 8y + 3
\end{align*} \]

\[ \begin{align*}
  &x \leftarrow x - 1 \\
  &y \leftarrow y - 1
\end{align*} \]
Reachability in affine $\mathbb{Z}$-VASS is undecidable

Reduction from the Post correspondence problem

$w_1 = \begin{array}{c} 10 \\ 1 \end{array}$  $w_2 = \begin{array}{c} 01 \\ 0 \end{array}$  $w_3 = \begin{array}{c} 1 \\ 011 \end{array}$

$x \leftarrow 4x + 2$
$y \leftarrow 2y + 1$

$x \leftarrow 4x + 1$
$y \leftarrow 2y$

$x \leftarrow 2x + 1$
$y \leftarrow 8y + 3$

$x \leftarrow x - 1$
$y \leftarrow y - 1$

p

q
Reachability in affine $\mathbb{Z}$-VASS is undecidable

Reduction from the Post correspondence problem

\[
\begin{align*}
 w_1 &= \frac{10}{1} \\
 w_2 &= \frac{01}{0} \\
 w_3 &= \frac{1}{011}
\end{align*}
\]

Has solution iff $p(1, 1) \xrightarrow{\mathbb{Z}} q(1, 1)$
Reachability in affine $\mathbb{Z}$-VASS is undecidable

Reduction from the Post correspondence problem

$$w_1 = \begin{array}{c} 10 \\ 1 \end{array} \quad w_2 = \begin{array}{c} 01 \\ 0 \end{array} \quad w_3 = \begin{array}{c} 1 \\ 011 \end{array}$$

Doubling can be done with a gadget of transfers and copies
Finite matrix monoids over $\mathbb{N}$
Finite matrix monoids over $\mathbb{N}$

- Infinite monoids
  - Affine $\mathbb{Z}$-VASS
  - Transfer $\mathbb{Z}$-VASS
  - Copy $\mathbb{Z}$-VASS
  - Reset $\mathbb{Z}$-VASS
  - Permutation $\mathbb{Z}$-VASS
- Finite monoids
  - Transfer + copy $\mathbb{Z}$-VASS
  - Reset + permutation $\mathbb{Z}$-VASS

$\in \text{PSPACE}$
Finite matrix monoids over $\mathbb{N}$

Some decidable

- Affine $\mathbb{Z}$-VASS
- Transfer + copy $\mathbb{Z}$-VASS
- Transfer $\mathbb{Z}$-VASS
- Copy $\mathbb{Z}$-VASS
- Reset + permutation $\mathbb{Z}$-VASS
- Reset $\mathbb{Z}$-VASS
- Permutation $\mathbb{Z}$-VASS
- $\mathbb{Z}$-VASS

$\in$ PSPACE
Conclusion: summary

• Unified approach to reachability in affine $\mathbb{Z}$-VASS

• Possible to remove transformations when matrix monoid is finite

• Reachability relation of affine $\mathbb{Z}$-VASS is semilinear when monoid is finite

• Classification of complexity w.r.t. extensions
Conclusion: further work

- Complexity of reachability for permutation $\mathbb{Z}$-VASS?

- Size of matrix monoid for arbitrary affine $\mathbb{Z}$-VASS?

- Characterization of classes of infinite matrix monoids for which reachability is decidable?
Thank you! Vielen Dank!