On the State Complexity of Population Protocols

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Joint work with Javier Esparza and Stefan Jaax

Technical University of Munich

Overview

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

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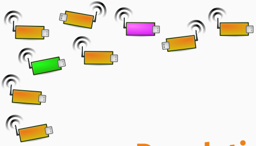

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

Can model *e.g.* networks of passively mobile sensors and chemical reaction networks

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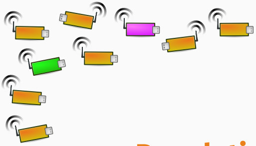

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Protocols compute predicates of the form $\varphi\colon \mathbb{N}^d \to \{0,1\}$

e.g. if φ is unary, then $\varphi(n)$ is computed by n agents

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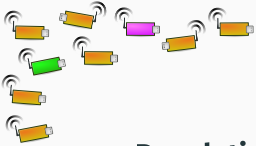

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

This talk: Study of the minimal size of protocols

Population protocols

Angluin *et al.* PODC'04

- anonymous mobile agents with very few resources
- agents change states via random pairwise interactions
- each agent has opinion true/false
- computes by stabilizing agents to some opinion

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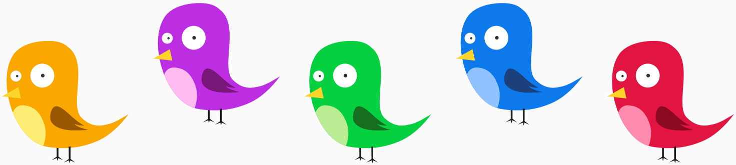

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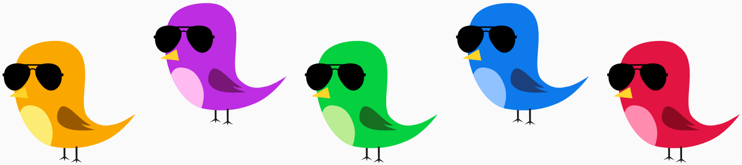

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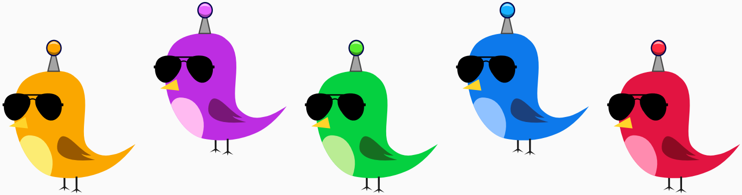

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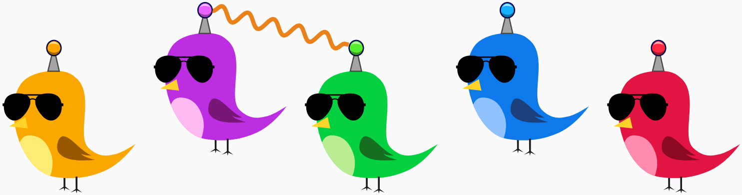

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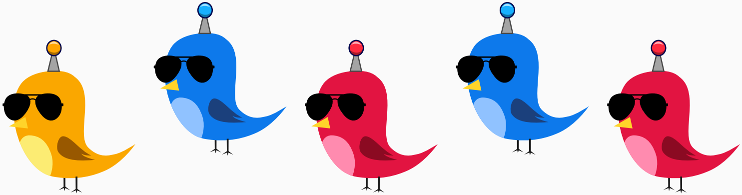

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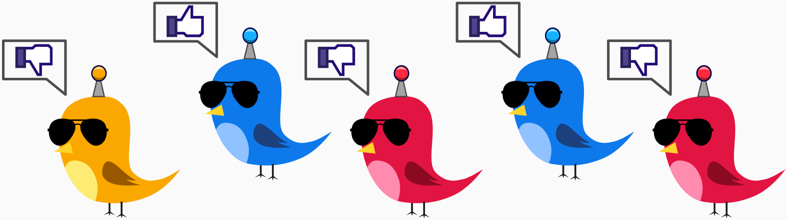

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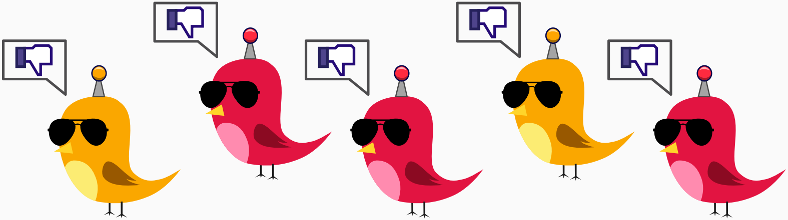

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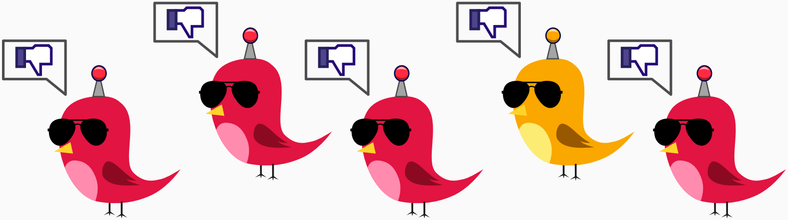

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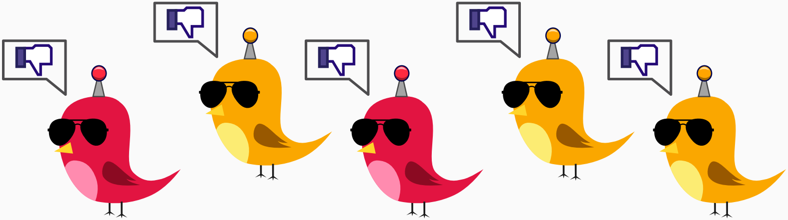
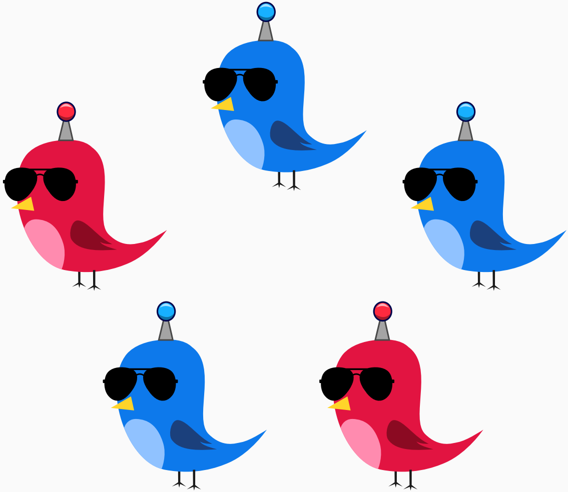


Example: majority protocol

More blue birds than red birds?



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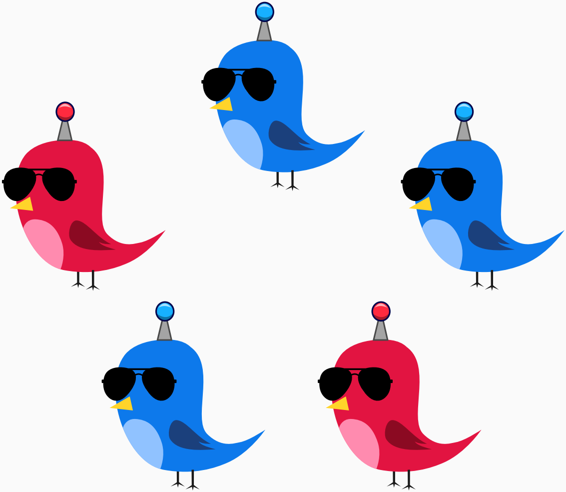
- Two large birds of different colors become small
- Large birds convert small birds to their color



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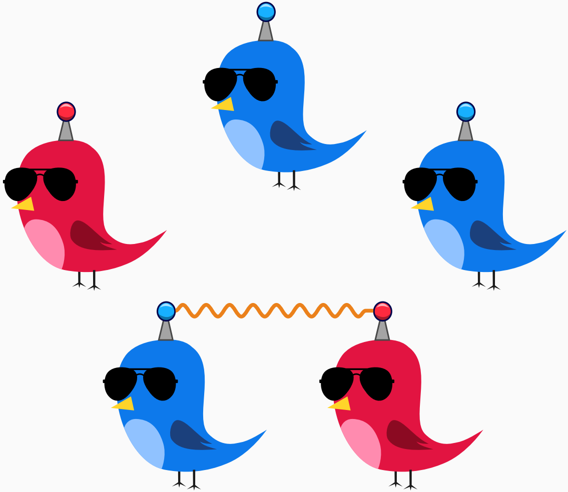

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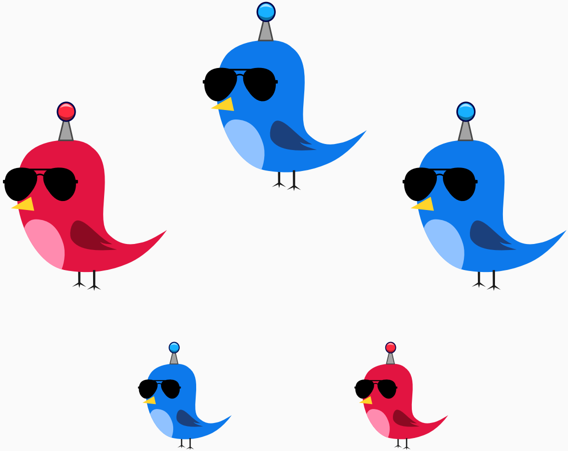

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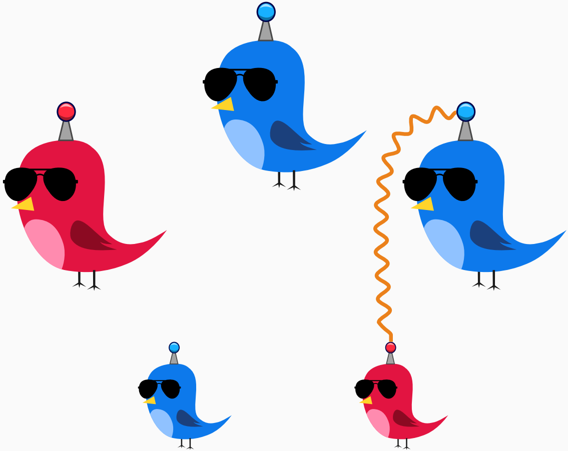

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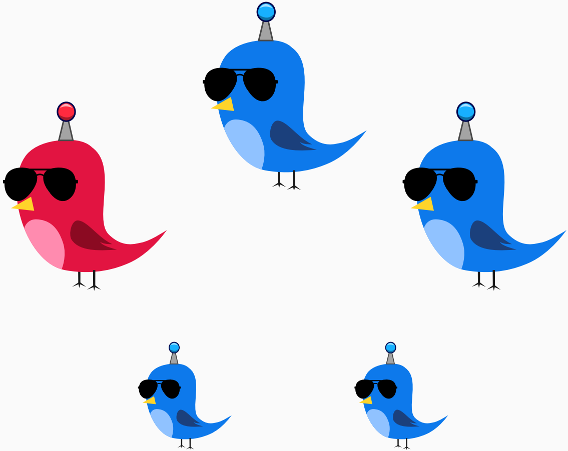

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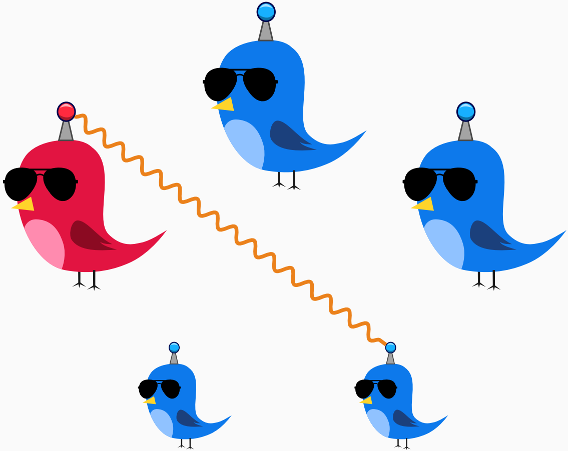

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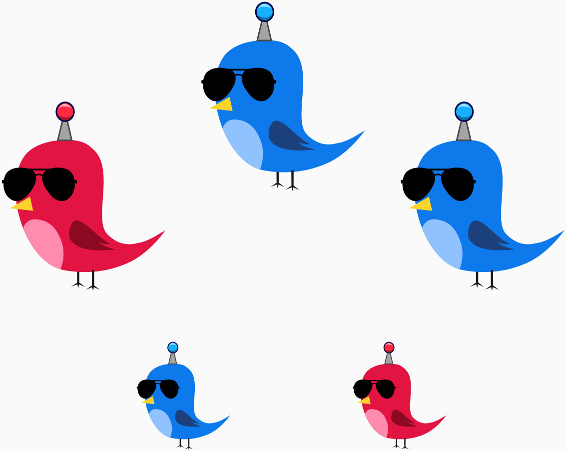

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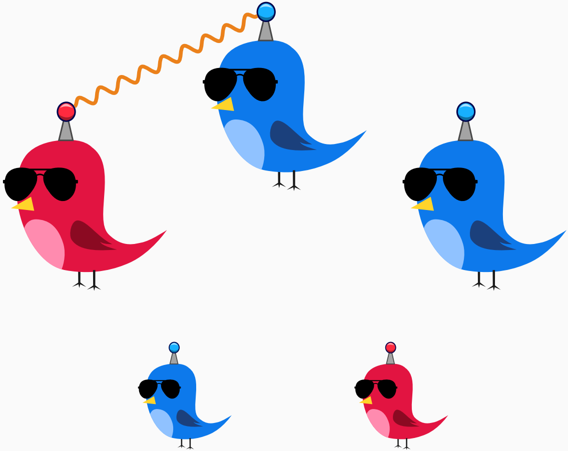

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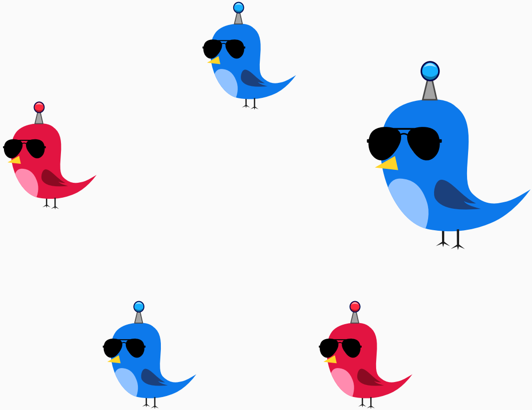

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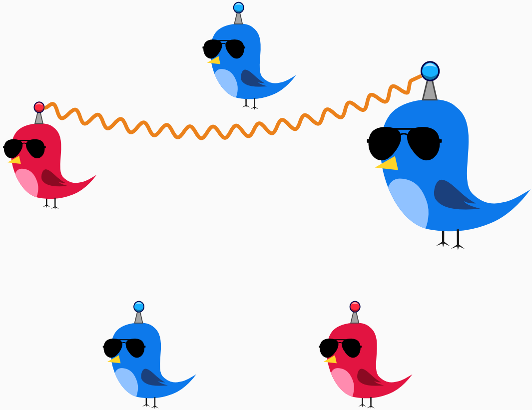

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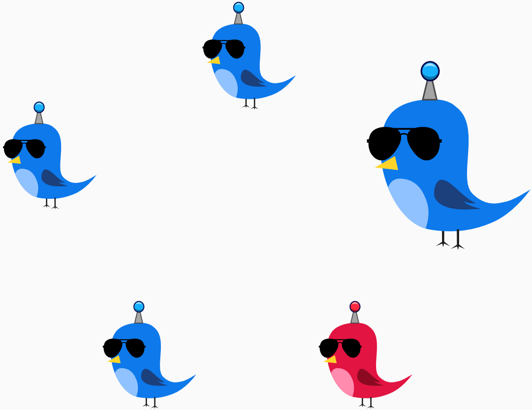

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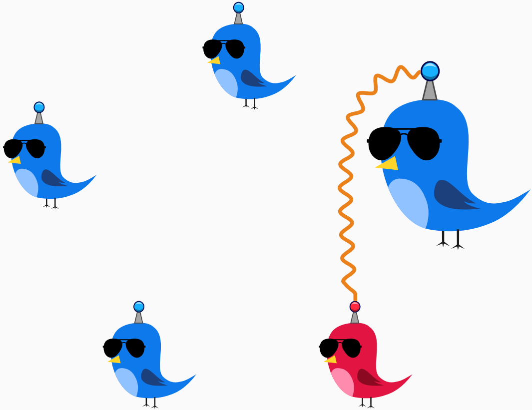

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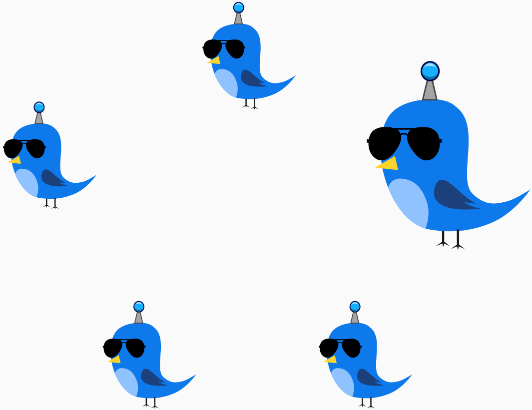

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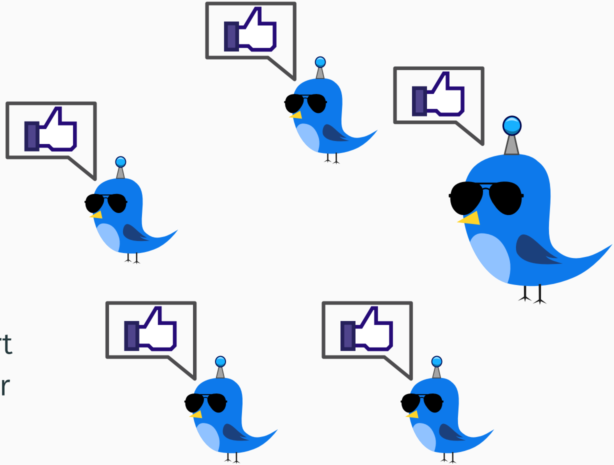

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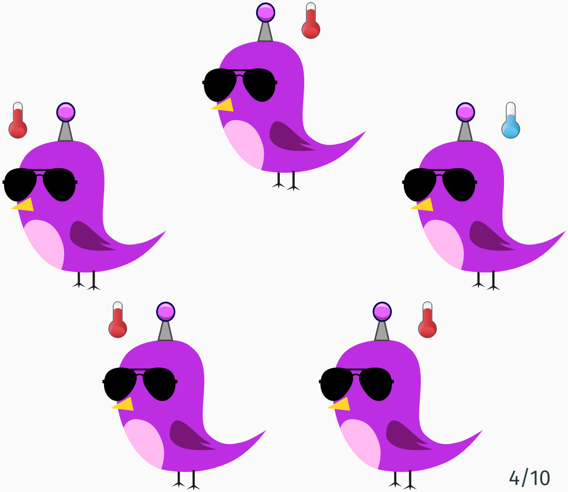
Are there at least 4 sick birds?



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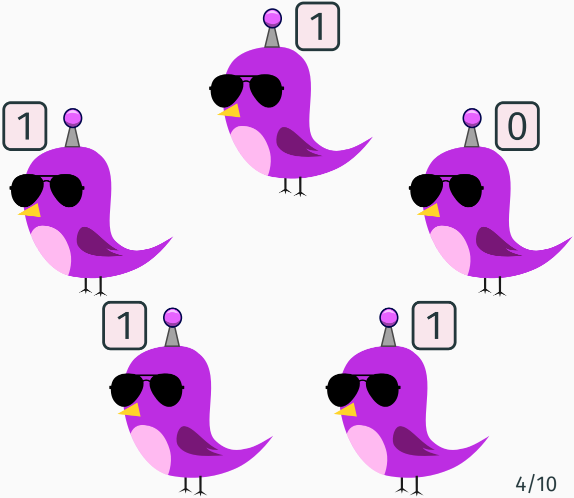

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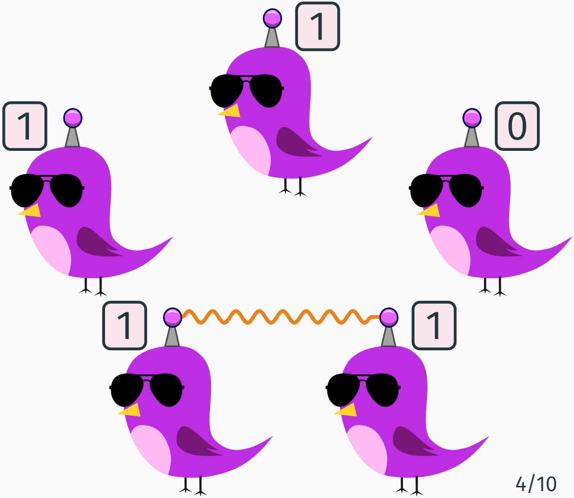

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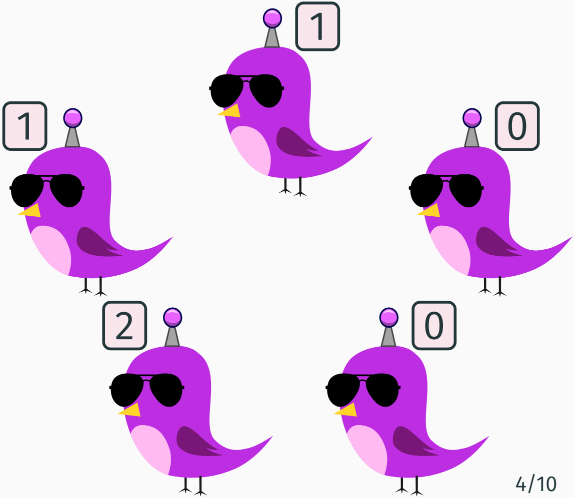

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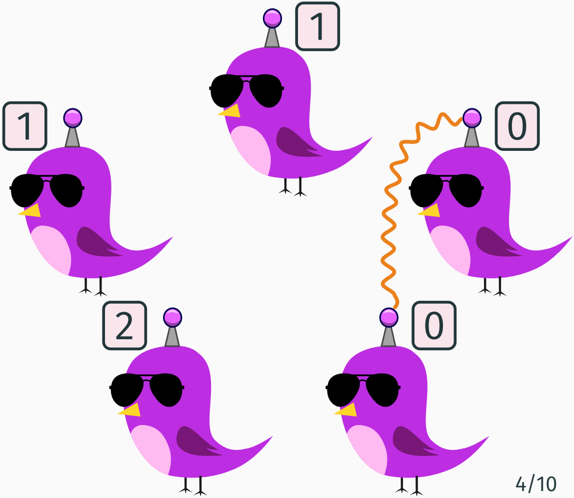

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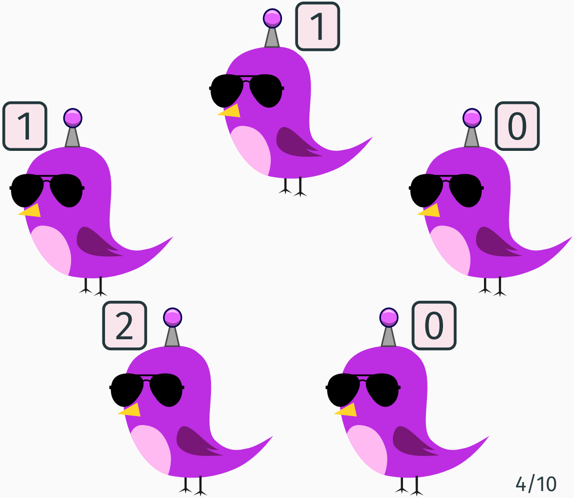

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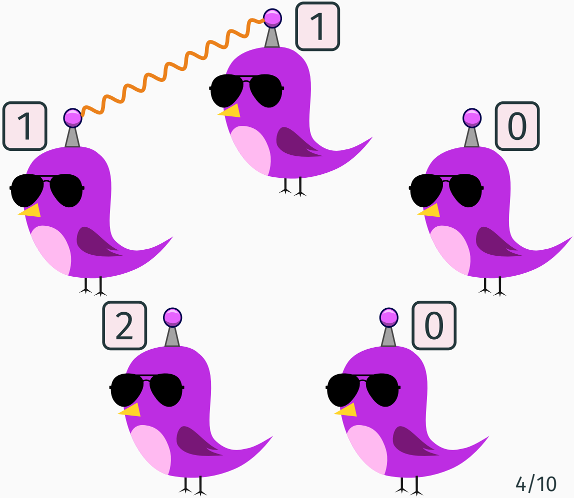

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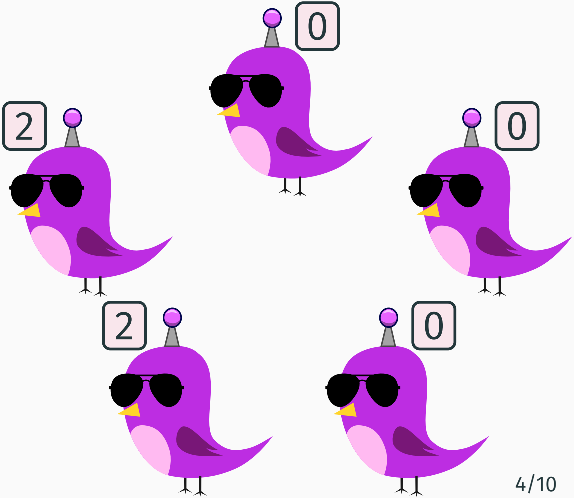

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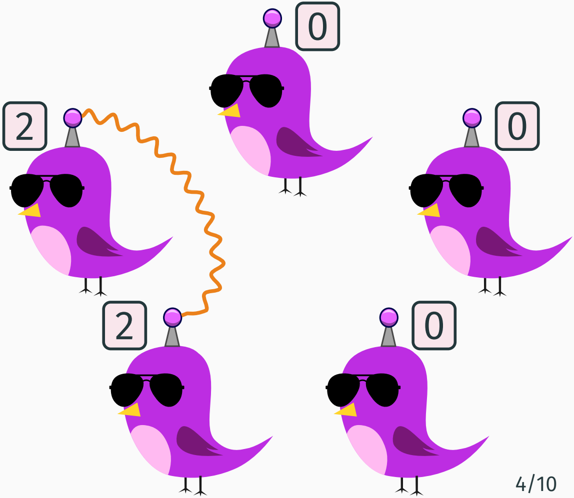

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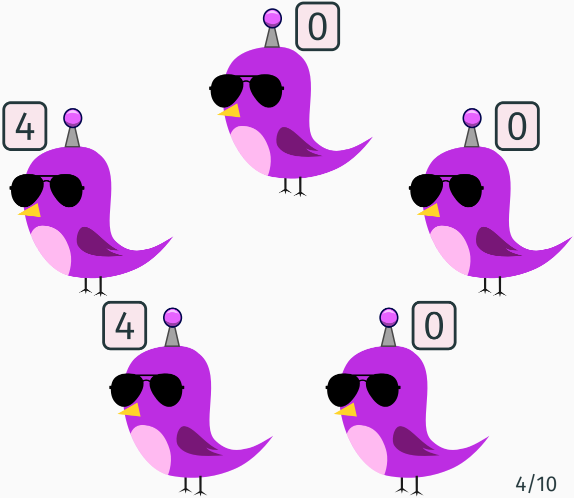

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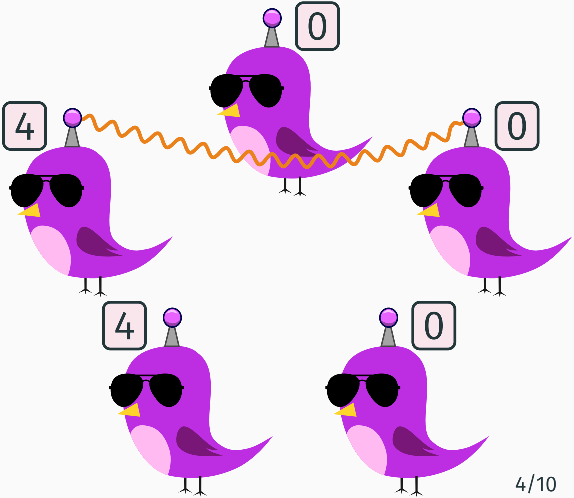

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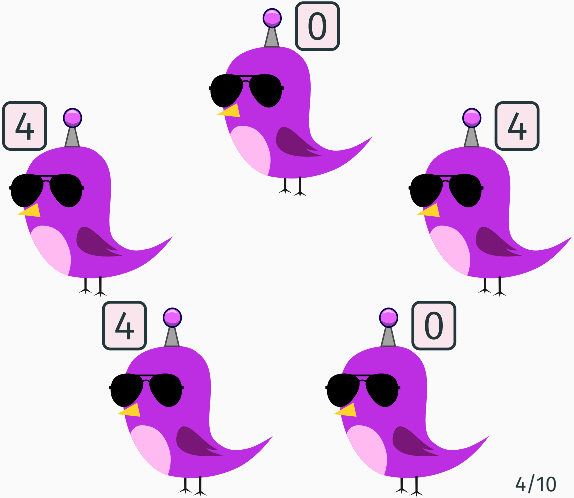

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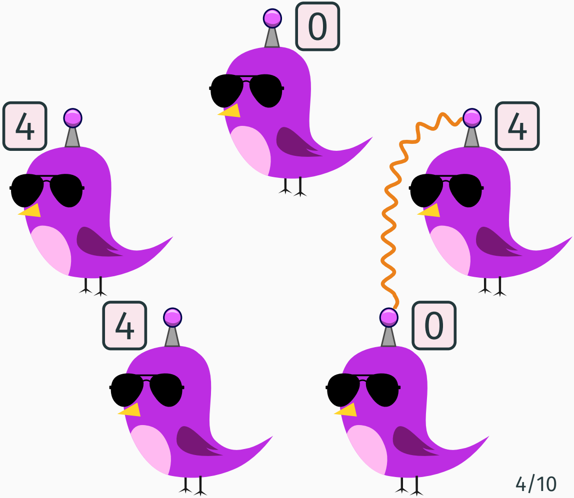

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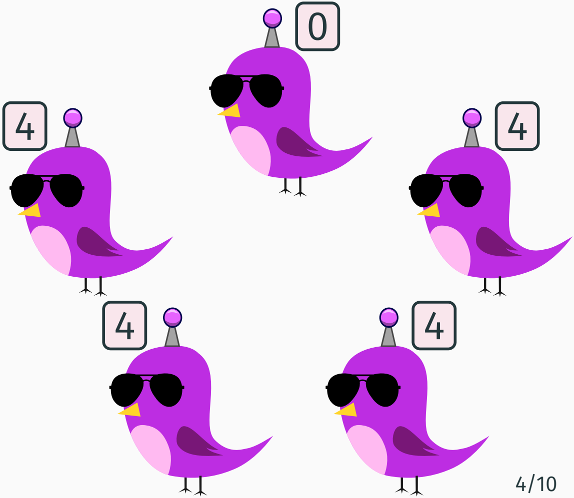

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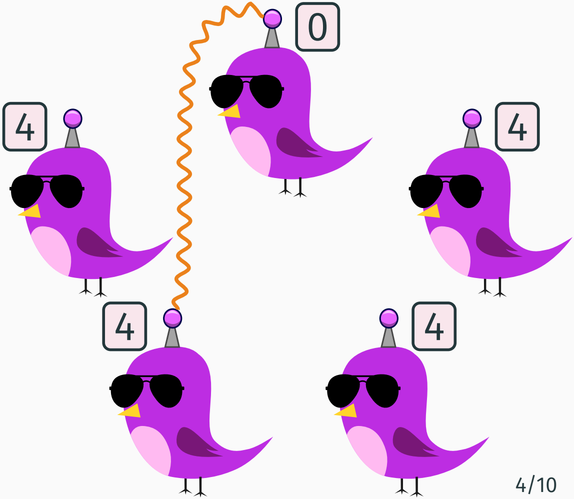

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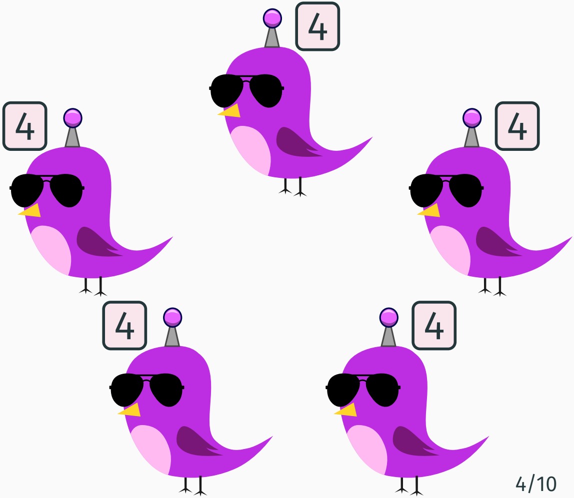

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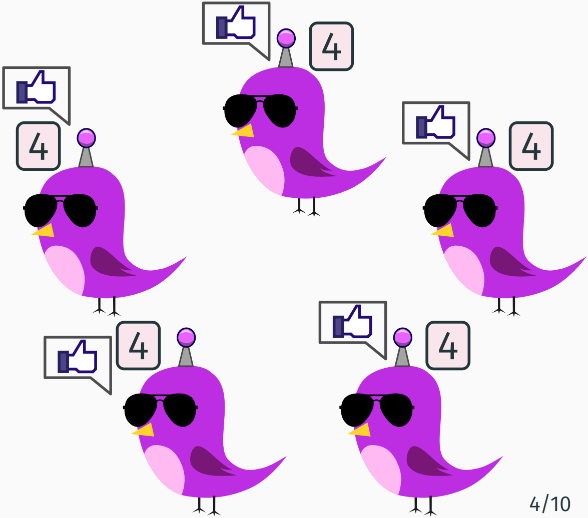
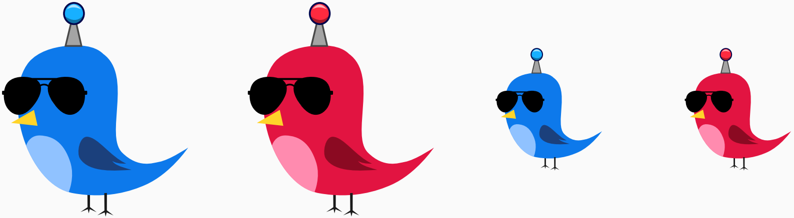


Demonstration

Population protocols: formal model

- *States*: finite set Q
- *Opinions*: $O : Q \to \{0, 1\}$
- *Initial states*: $I \subseteq Q$
- *Transitions*: $T \subseteq Q^2 \times Q^2$



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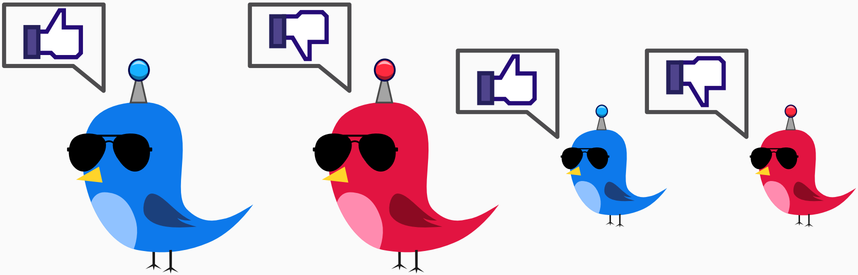

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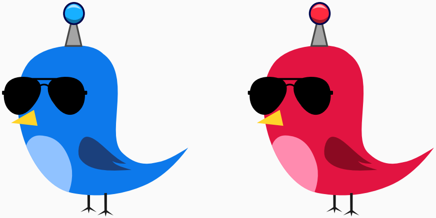

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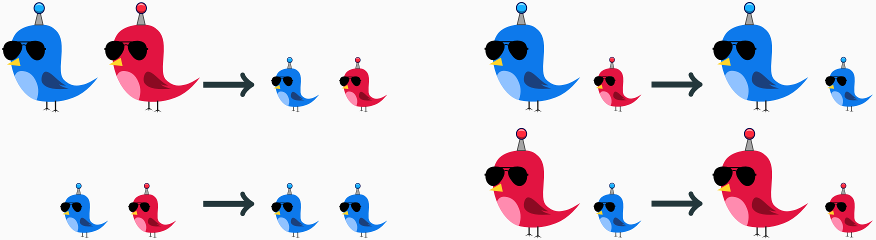
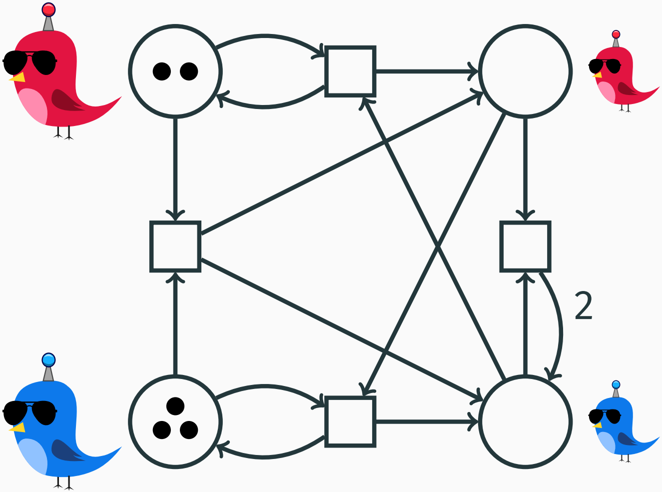
- *Opinions*: $O : Q \to \{0, 1\}$
- *Initial states*: $I \subseteq Q$
- *Transitions*: $T \subseteq Q^2 \times Q^2$



Population protocols: formal model

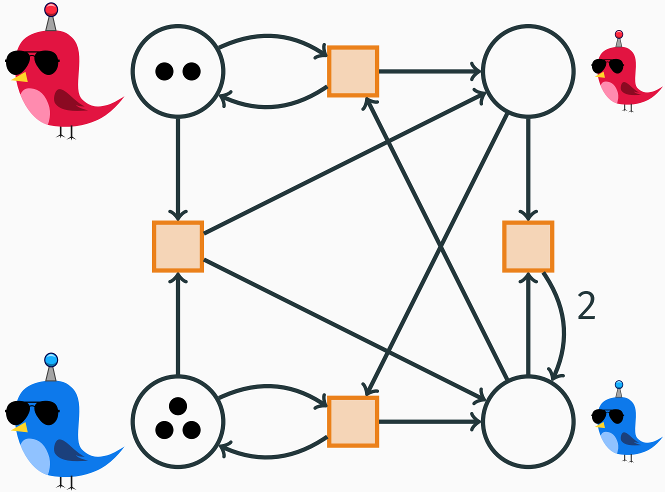
Protocols can be translated into Petri nets



Population protocols: formal model

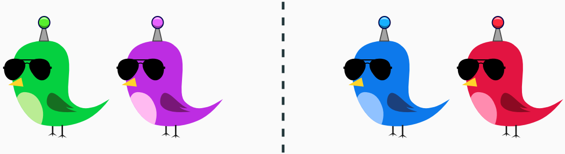
Protocols can be translated into Petri nets

conservative / bounded



Population protocols: initial configurations

Initial configurations $= L + \mathbb{N}^I$ for some $L \in \mathbb{N}^Q$



Leaders L Arbitrary number of initial states

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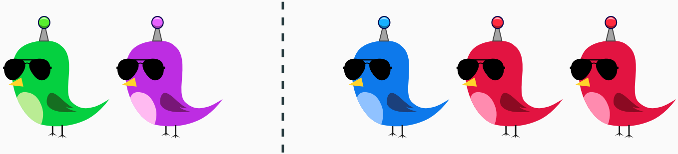

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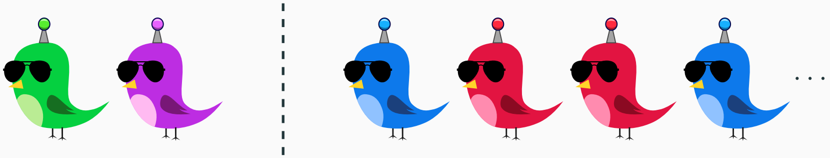
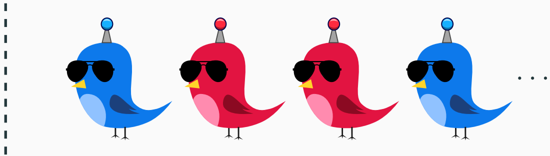


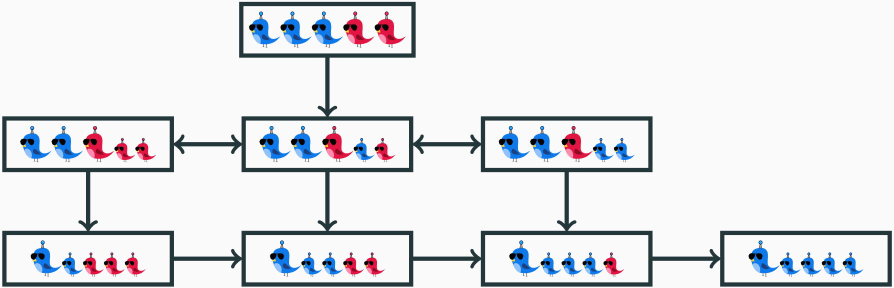
Leaders L Arbitrary number of initial states

Initial configurations $= \mathbf{0} + \mathbb{N}^I$

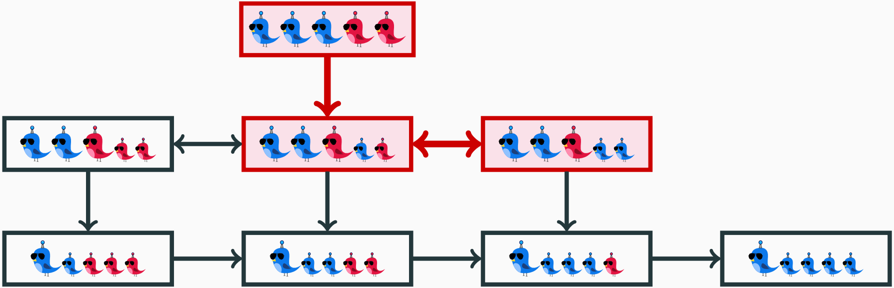


No leaders in protocols seen so far

Reachability graph:



Executions must be fair:



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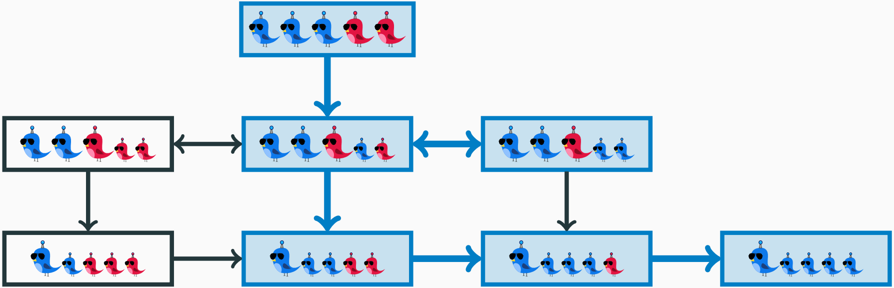
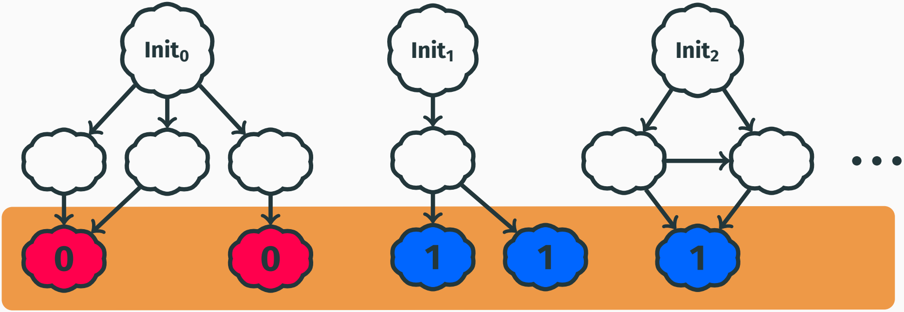


A protocol computes a predicate $f\colon \text{Init} \to \{0, 1\}$

if fair executions reach common **consensus**



A protocol computes a predicate $f\colon \mathbf{Init} \to \{0, 1\}$

if fair executions reach common **consensus**

Expressive power **Angluin, Aspnes, Eisenstat PODC'06**

Population protocols compute precisely predicates definable in Presburger arithmetic, *i.e.* FO($\mathbb{N}$, $+$, $<$)

State complexity

Number of states corresponds to amount of memory, so relevant to keep it small for embedded systems

Protocol size also crucial for verification

- **B** $\geq$ **R** requires at least 4 states (Mertzios *et al.* ICALP'14)
- **X** $\geq$ **c** requires at most c + 1 states

State complexity

Given: Presburger-definable predicate φ

Question: Smallest number of states necessary to compute φ?

State complexity

Given: Presburger-definable predicate φ

Question: Smallest number of states necessary to compute φ?

Difficult problem...

What about basic predicates?

State complexity: threshold

Given: $c \in \mathbb{N}$

Question: Smallest number of states necessary to compute $x \geq c$?

State complexity: threshold

Given: $c \in \mathbb{N}$

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Upper bound: $c + 1$

Lower bound: 2

State complexity: threshold

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Question: Smallest number of states necessary to compute $x \geq c$?

Upper bound: $c + 1$

Lower bound: 2

Theorem STACS'18

$x \geq c$ is computable with $O(\log c)$ states, if $c = 2^n$.

Proof sketch

States:

$\{0, 2^0, 2^1, \ldots, c\}$

Output:

$O(m) = 1 \Leftrightarrow m = c$

Rules:

$$\begin{array}{ccll}
(1,1) & \mapsto & (2,0) & \\
(2,2) & \mapsto & (4,0) & \\
\vdots & & \vdots & \\
(2^{n-1}, 2^{n-1}) & \mapsto & (2^n, 0) & \\
(m,n) & \mapsto & (c,c) & \text{if } m+n \geq c
\end{array}$$

State complexity: threshold

Given: $c \in \mathbb{N}$

Question: Smallest number of states necessary to compute $x \geq c$?

Upper bound: $c + 1$

Lower bound: 2

Theorem STACS'18

$x \geq c$ is computable with $O(\log c)$ states, if ~~$c = 2^n$~~.

Proof sketch

may fail if c is not a power of 2

States:
$\{0, 2^0, 2^1, \ldots, c\}$

Output:
$O(m) = 1 \Leftrightarrow m = c$

Rules:

$$\begin{aligned}
(1,1) &\mapsto (2,0) \\
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Proof sketch

Erroneous run for $c = 7$:

$$\{1, 1, 1, 1, 1, 1, 1\}$$
$$\downarrow *$$
$$\{2, 0, 2, 0, 2, 0, 1\}$$
$$\downarrow$$
$$\{4, 0, 0, 0, 2, 0, 1\}$$

State complexity: threshold

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Upper bound: $c + 1$

Lower bound: 2

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$x \geq c$ is computable with $O(\log c)$ states, if $c = 2^n$.

Solution:

Add a few extra states

Proof sketch

States:

$\{0, 1, 2, 4, 6, 7\}$

Output:

$O(m) = 1 \Leftrightarrow m = 7$

Rules:

$$
\begin{aligned}
(1,1) &\mapsto (2,0) \\
(2,2) &\mapsto (4,0) \\
(4,2) &\mapsto (6,0) \\
(4,4) &\mapsto (7,7) \\
(6,m) &\mapsto (7,7)
\end{aligned}
$$

State complexity: threshold

Given: $c \in \mathbb{N}$

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Upper bound: $O(\log c)$

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Upper bound: $O(\log c)$

Lower bound: 2

Theorem **STACS'18**

Let $P_0, P_1, \ldots$ be protocols such that P_c computes $x \geq c$.
There are infinitely many c s.t. P_c has $\geq (\log c)^{1/4}$ states.

Proof sketch

Counting argument on # states vs. # unary predicates

State complexity: threshold

Given: $c \in \mathbb{N}$

Question: Smallest number of states necessary to compute $x \geq c$?

Upper bound: $O(\log c)$

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Possible to go below log c for some c?

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Lower bound: $\underbrace{O(\log^{1/4} c)}_{\text{for inf. many c}}$

Possible to go below log c for some c?

Yes, with few leaders!

Threshold: sublogarithmic upper bound

Theorem **STACS'18**

There exist protocols $P_0, P_1, \ldots$ and numbers $c_0 < c_1 < \cdots$ s.t. P_i computes $x \geq c_i$ and has $O(\log \log c_i)$ states and 2 leaders.

Threshold: sublogarithmic upper bound

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Lemma **Mayr and Meyer '82**

For every $c \in \mathbb{N}$, there exists a reversible multiset rewriting system $\mathcal{R}_c$ over alphabet $\Sigma \supseteq \{x, y, z, \bigstar\}$ of size $O(c)$ with rewriting rules $T \subseteq \Sigma^{\leq 5} \times \Sigma^{\leq 5}$ such that

$$\{x, y\} \xrightarrow{*} M \text{ and } \bigstar \in M \iff M = \{y, z^{2^{2^c}}, \bigstar\}$$

Threshold: sublogarithmic upper bound

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Proof sketch

- $\mathcal{R}_c$ can be simulated by adding a padding symbol $\perp$

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- $\mathcal{R}_c$ can be simulated by adding a padding symbol $\perp$

Rewriting system $\mathcal{R}_c$	5-way population protocol
$(e, f, g) \mapsto (h, i)$	$(e, f, g, \perp, \perp) \mapsto (h, i, \perp, \perp, \perp)$
$(e, f) \mapsto (g, h, i)$	$(e, f, \perp, \perp, \perp) \mapsto (g, h, i, \perp, \perp)$

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Each 5-way transition is converted to
a "gadget" of 2-way transitions

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Proof sketch

- $\mathcal{R}_c$ can be simulated by adding a padding symbol $\perp$
- New rule: agents in state $\star$ can convert others to $\star$

Threshold: sublogarithmic upper bound

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Proof sketch

- $\mathcal{R}_c$ can be simulated by adding a padding symbol $\bot$
- New rule: agents in state $\bigstar$ can convert others to $\bigstar$
- Simulate $\mathcal{R}_c$ from $\{x, y, \bot, \bot, \ldots, \bot\}$

Threshold: sublogarithmic upper bound

Theorem **STACS'18**

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Proof sketch

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- New rule: agents in state $\bigstar$ can convert others to $\bigstar$
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- $\{\bigstar, \bigstar, \ldots, \bigstar\}$ reachable $\iff$ initially $\geq 2^{2^c}$ agents in $\perp$

Threshold: sublogarithmic upper bound

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- $\mathcal{R}_c$ can be simulated by adding a padding symbol $\bot$
- New rule: agents in state $\bigstar$ can convert others to $\bigstar$
- Simulate $\mathcal{R}_c$ from $\{x, y, \bot, \bot, \ldots, \bot\}$
- $\{\bigstar, \bigstar, \ldots, \bigstar\}$ reachable $\iff$ initially $\geq 2^{2^c}$ agents in $\bot$
- By reversibility and fairness, cannot avoid $\{\bigstar, \bigstar, \ldots, \bigstar\}$

Threshold: lower bounds for 1-aware protocols

A protocol is *1-aware* if there is a subset of states Q_1 such that for every fair execution $\pi = C_0 C_1 \cdots$

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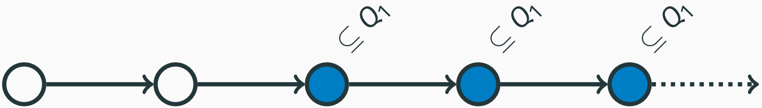
(a) if π stabilizes to 1, then there is some i such that $\mathrm{states}(C_j) \subseteq Q_1$ for every $j \geq i$



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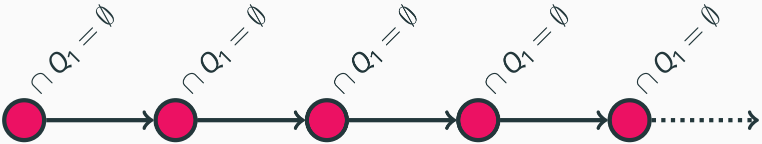

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Observation

1-aware protocols compute monotonic Presburger-definable protocols, including $x \geq c$

Threshold: lower bounds for 1-aware protocols

Theorem **STACS'18**

Every 1-aware protocol $\mathcal{P}$ computing $x \geq c$ has at least

(a) $\log_3 c$ states, if $\mathcal{P}$ is leaderless

(b) $(\log\log(c)/151)^{1/9}$ states, otherwise

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(b) $(\log\log(c)/151)^{1/9}$ states, otherwise

Proof sketch

- $\{c \cdot q_0\} \xrightarrow{*} D$ with $\mathrm{states}(D) \cap Q_1 \neq \emptyset$
- there exists m and D' s.t.

$$\{m \cdot q_0\} \xrightarrow{*} D',\ m \leq 3^{|Q|} \text{ and } \mathrm{states}(D) \subseteq \mathrm{states}(D')$$

- Thus, $c \leq m \leq 3^{|Q|}$ and hence $\log_3 c \leq |Q|$

Threshold: lower bounds for 1-aware protocols

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Every 1-aware protocol $\mathcal{P}$ computing $x \geq c$ has at least

(a) $\log_3 c$ states, if $\mathcal{P}$ is leaderless

(b) $(\log \log(c)/151)^{1/9}$ states, otherwise

Proof sketch

By Rackoff TCS'78:

If a state $q \in Q_1$ is coverable from $C \in \text{Init}$, then
q is coverable from C by an execution of length at most $2^{2^{\text{poly}(n)}}$

Linear inequalities

Let $A \in \mathbb{Z}^{m \times k}$, let $\boldsymbol{c} \in \mathbb{Z}^m$ and let n be the largest absolute value of numbers occurring in A and $\boldsymbol{c}$.

Observation

Classical protocol computing $A\boldsymbol{x} + \mathbf{c} > \mathbf{0}$ has $O(n^m)$ states.

Linear inequalities

Let $A \in \mathbb{Z}^{m \times k}$, let $\boldsymbol{c} \in \mathbb{Z}^m$ and let n be the largest absolute value of numbers occurring in A and $\boldsymbol{c}$.

Observation

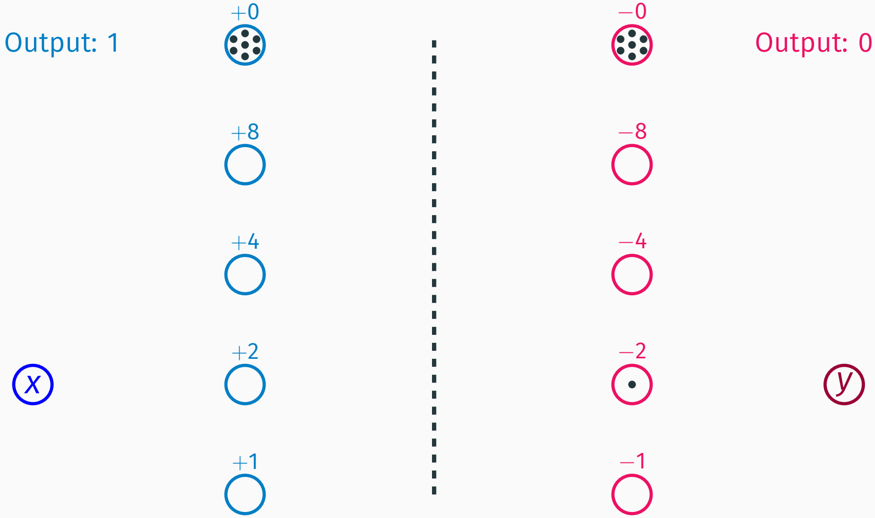
Classical protocol computing $A\boldsymbol{x} + \boldsymbol{c} > \mathbf{0}$ has $O(n^m)$ states.

Theorem **STACS'18**

There exists a protocol that computes $A\boldsymbol{x} + \boldsymbol{c} > \mathbf{0}$ and has

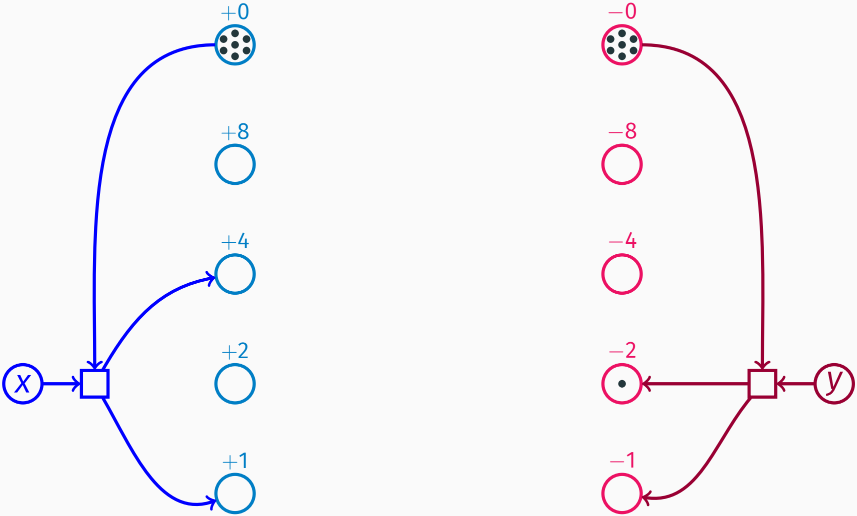
- at most $O((m + k) \cdot \log mn)$ states
- at most $O(m \cdot \log mn)$ leaders

Linear inequalities: example



Protocol for $5x - 3y > 2$

Linear inequalities: example



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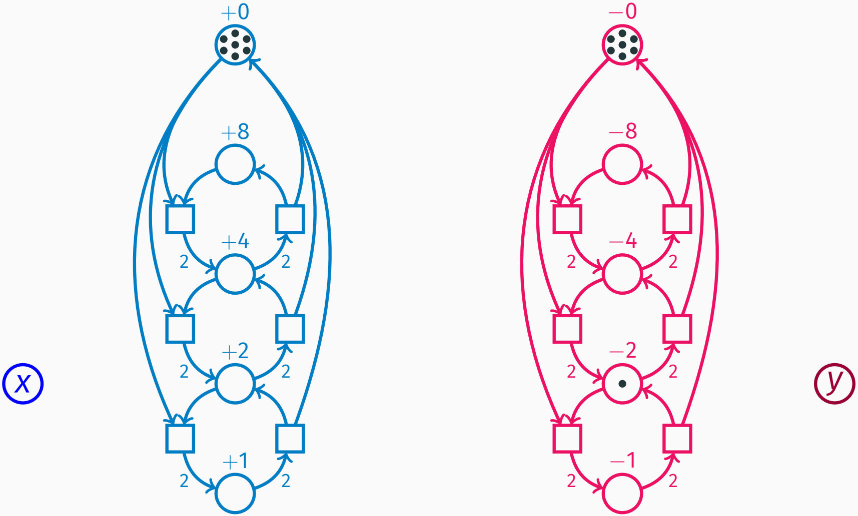

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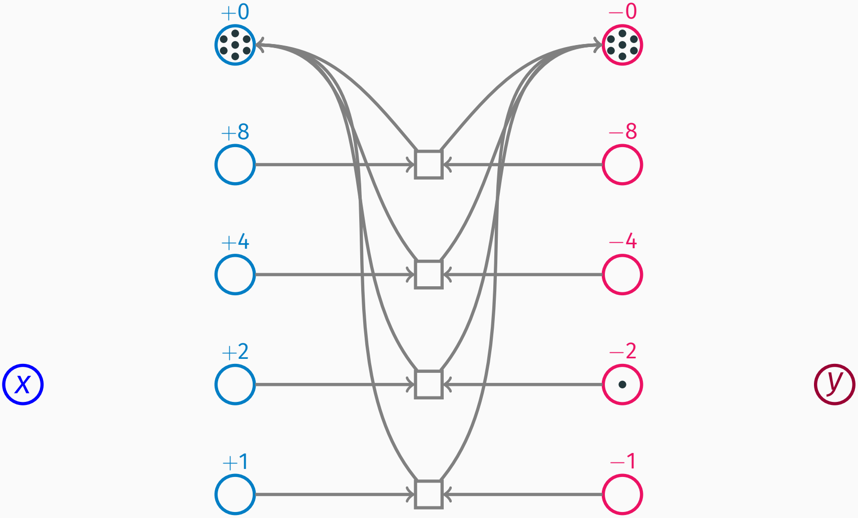

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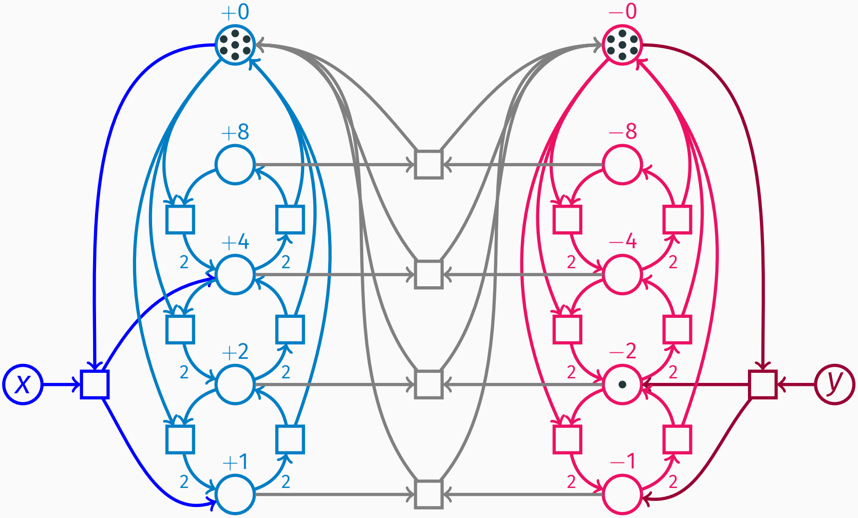

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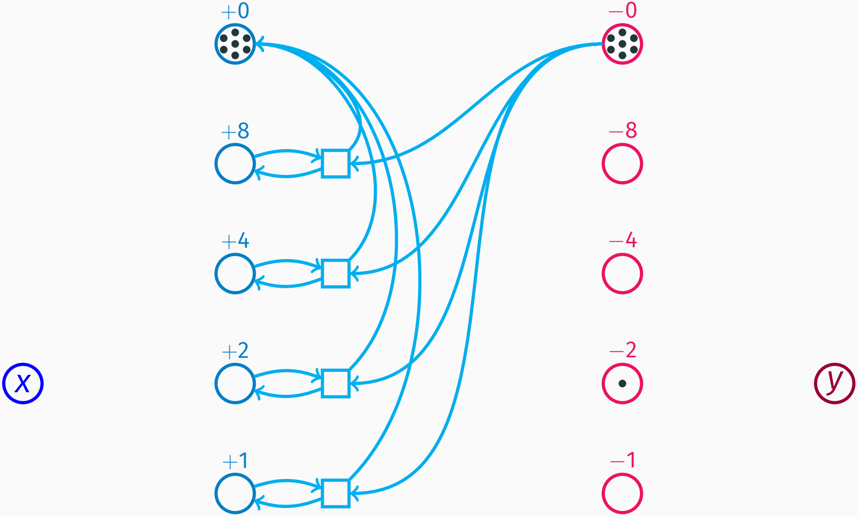

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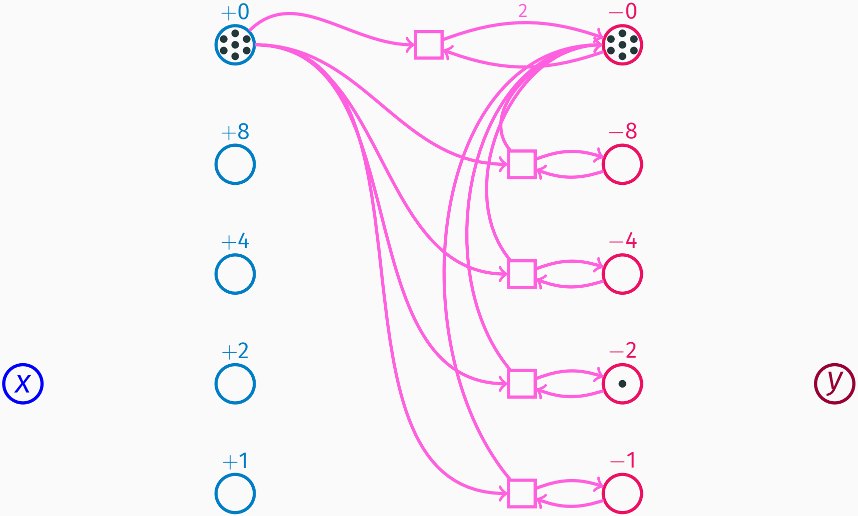

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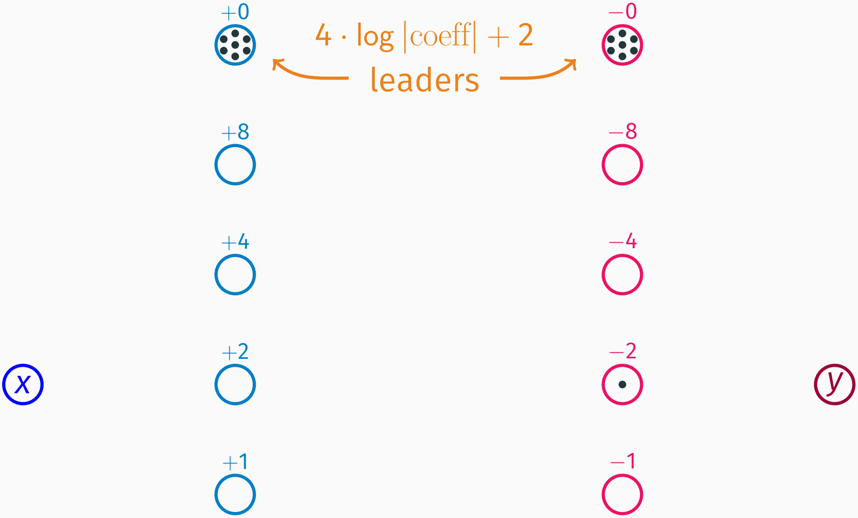

Protocol for $5x - 3y > 2$

- Complexity of $x \geq c$ can be decreased from $O(c)$ to $O(\log c)$ and sometimes $O(\log\log c)$
- Matching lower bounds for the class of 1-aware protocols
- Better upper bounds for systems of linear inequalities

Conclusion: future work

- Is $O(\log\log\log c)$ states sometimes possible for computing $x \geq c$?
- State complexity of more general Presburger-definable predicates?
- Study of the trade-off between size and speed

Thank you!