Handling Infinite Branching WSTS

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- Moreover, multiple decidability results are known on WSTS.
- However, most results and techniques known suppose finite branching.
- We propose a tool, the WSTS *completion*, based on work of Finkel and Goubault-Larrecq, to handle infinitely branching WSTS.

Ordered transition system

$S = (X, \to, \leq)$ where

- X set,
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$S = (X, \rightarrow, \leq)$ where

- X set: recursively enumerable,
- $\rightarrow \subseteq X \times X$: decidable,
- $\leq$ quasi-ordering X: decidable.

Well-ordered transition system (WSTS)

A *WSTS* is an ordered transition system $(X, \to, \leq)$ with

- well-quasi-ordering: $\forall x_0, x_1, \ldots \; \exists i < j \;$ s.t. $\; x_i \leq x_j$,
- monotony:

$$
\begin{array}{cccc}
\forall \; x & \to & y & \\
\wedge\!\!\!| & & \wedge\!\!\!| & \\
x' & \xrightarrow{*} & y' & \exists
\end{array}
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- transitive monotony:

$$\begin{array}{cccc} \forall\ x & \rightarrow & y & \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & \\ x' & \xrightarrow{+} & y' & \exists \end{array}$$

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- Essentially finite WSTS (Abdulla, Cerans, Jonsson & Tsay 2000),

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- Essentially finite WSTS (Abdulla, Cerans, Jonsson & Tsay 2000),
- Do you know other ones?

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A tool

Develop from the WSTS *completion* introduced by Finkel & Goubault-Larrecq 2009.

Ideals

$I \subseteq X$ is an *ideal* if it is

- downward closed: $I = \downarrow I$,
- directed: $a, b \in I \implies \exists c \in I$ s.t. $a \leq c$ and $b \leq c$.

Theorem (Finkel & Goubault-Larrecq 2009; GL 2014)

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Theorem (Finkel & Goubault-Larrecq 2009; GL 2014; BFM 2014)

Every downward closed subset decomposes canonically as the union of its maximal ideals.

Completion (Finkel & Goubault-Larrecq 2009; BFM 2014)

The *completion* of $S = (X, \rightarrow_S, \leq)$ is $\widehat{S} = (\widehat{X}, \rightarrow_{\widehat{S}}, \subseteq)$ such that

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- $\widehat{X} = \mathrm{Ideals}(X)$,
- $I \rightarrow_{\widehat{S}} J$ if $\downarrow \mathrm{Post}(I) = \underbrace{\ldots \cup J \cup \ldots}_{\text{canonical}}$

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Relating executions of S and $\widehat{S}$

Let $S = (X, \rightarrow_S, \leq)$ be a WSTS, then

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- if $I \xrightarrow{k}_{\widehat{S}} J$, then for every $y \in J$ there exists $x \in I$ such that $x \xrightarrow{*}_S y' \geq y$.

Relating executions of S and $\widehat{S}$

Let $S = (X, \rightarrow_S, \leq)$ be a WSTS with transitive monotony, then

- if $x \xrightarrow{k}_S y$, then for every ideal $I \supseteq \downarrow x$ there exists an ideal $J \supseteq \downarrow y$ such that $I \xrightarrow{k}_{\widehat{S}} J$,
- if $I \xrightarrow{k}_{\widehat{S}} J$, then for every $y \in J$ there exists $x \in I$ such that $x \xrightarrow{\geq k}_S y' \geq y$.

Termination

Input: $(X, \to, \leq)$ a WSTS, $x_0 \in X$.

Question: $\nexists x_0 \to x_1 \to x_2 \to \ldots$?

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Proposition (Dufourd, Jančar & Schnoebelen 1999)

Termination is undecidable for infinitely branching WSTS.

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Input: $(X, \to, \leq)$ a WSTS, $x_0 \in X$.

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Remark

Strong termination and termination are the same in finitely branching WSTS.

Theorem (Blondin, Finkel & McKenzie 2014)

Strong termination is decidable for WSTS with transitive monotony and such that $\widehat{S}$ is a post-effective WSTS.

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Proof

Executions bounded in S iff bounded in $\widehat{S}$.

Theorem (Blondin, Finkel & McKenzie 2014)

Strong termination is decidable for WSTS with transitive monotony and such that $\widehat{S}$ is a post-effective WSTS.

Proof

Executions bounded in S iff bounded in $\widehat{S}$. Since $\widehat{S}$ finitely branching, we can decide termination in $\widehat{S}$ by Finkel & Schnoebelen 2001.

Coverability

Input: $(X, \rightarrow, \leq)$ a WSTS, $x_0, x \in X$.

Question: $x_0 \xrightarrow{*} x' \geq x$?

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Backward method (Abdulla, Cerans, Jonsson & Tsay 2000)

Compute $Y_0, \ldots, Y_n$ converging to $\uparrow \mathrm{Pre}^*(\uparrow x)$ and verify if $x_0 \in Y_n$.

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Forward method

Coverability:

- Enumerate executions $\downarrow x_0 \xrightarrow{*}_{\widehat{S}} I$,
- Accept if $x \in I$.

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- Accept if $x \in I$.

Non coverability:

- Enumerate $D \subseteq X$ downward closed, $x_0 \in D$ and $\downarrow\mathrm{Post}_S(D) \subseteq D$,
- Reject if $x \notin D$.

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- Enumerate $D = I_1 \cup \ldots \cup I_k$
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- **What other applications has the completion?**
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- Algorithms working on the completion more efficient for what WSTS/problems?

Thank you!